Math 531 - Partial Differential Equations
Heat Conduction —
in Higher Dimensions

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Spring 2020
Outline

1. Heat Equation 3D
   - Derivation
   - Heat Equation

2. Laplacian in Other Coordinates
   - Poisson’s and Laplace’s Equations
   - Other Coordinates
Previously we developed the *heat equation* for a one-dimensional rod.

We want to extend the *heat equation* for higher dimensions.

**Conservation of Heat Energy:** In any volume element, the basic conservation equation for heat satisfies

\[
\text{Rate of change of heat energy in time} = \text{Heat energy flowing across boundaries per unit time} + \text{Heat energy generated inside per unit time}
\]

Define \(c(x, y, z)\) to be the *specific heat of a material* (the *heat energy* required to raise a unit mass of a material a unit of temperature).

Define \(\rho(x, y, z)\) to be the *mass density* (per unit volume).

Define \(u(x, y, z, t)\) as the *temperature of a material*.
The specific heat, \( c \), mass density, \( \rho \), and temperature, \( u \), are used with the conservation law above to create the general heat equation. The total energy in a volume element \( R \) satisfies:

\[
\text{Total Energy} = \iiint_R c \rho u \, dV.
\]

The rate of change of heat energy in time is given by

\[
\frac{d}{dt} \iiint_R c \rho u \, dV.
\]
Heat Conduction in a Higher Dimensions

Define $\phi(x, y, z, t)$ as the *heat flux vector* for the heat crossing the surface of the region $R$ denoted $\partial R$, and define $n$ as the *outward normal vector*

By convention the heat flux is the flow directed into the region $R$, so the *heat flux* into the region $R$ is the integral over $\partial R$ of $-\phi \cdot n$.

$$-\iint_{\partial R} \phi \cdot n \, dS$$
Define $Q(x, y, z, t)$ as the *heat energy* generated per unit time from the *sources* or *sinks* inside the region $R$.

This gives

$$
\iiint_{R} Q \, dV.
$$

The *Conservation of Heat Energy* combines these terms to give:

$$
\frac{d}{dt} \iiint_{R} c \rho u \, dV = - \iiint_{\partial R} \phi \cdot n \, dS + \iiint_{R} Q \, dV.
$$

We need to combine these terms to obtain the general *Heat Equation*. 
Theorem (Divergence or Gauss’s Theorem)

Suppose \( R \) is a subset of \( \mathbb{R}^3 \), which is compact and has a piecewise smooth boundary \( \partial R \). If \( \phi \) is a continuously differentiable vector field defined on a neighborhood of \( R \), then we have:

\[
\iint_{\partial R} (\phi \cdot n) \, dS = \iiint_{R} (\nabla \cdot \phi) \, dV.
\]

The Conservation of Heat Energy combines these terms to give:

\[
\frac{d}{dt} \iiint_{R} c\rho u \, dV = - \iiint_{R} (\nabla \cdot \phi) \, dV + \iiint_{R} Q \, dV.
\]
Heat Conduction in a Higher Dimensions

The previous equation is rearranged to give:

$$\iiint_{R} \left( c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q \right) \, dV = 0.$$  

Since this holds for any region $R$, we have the heat equation:

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \phi + Q.$$  

Fourier’s law of heat conduction satisfies:

$$\phi = -K_0 \nabla u,$$

which produces the heat equation in higher dimensions:

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q.$$
The **heat equation** in higher dimensions is:

\[ c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q. \]

If the Fourier coefficient is constant, \( K_0 \), as well as the specific heat, \( c \), and material density, \( \rho \), and if there are no sources or sinks, \( Q \equiv 0 \), then the **heat equation** becomes

\[ \frac{\partial u}{\partial t} = k \nabla^2 u, \quad t > 0 \quad \text{and} \quad (x, y, z) \in R, \]

where \( k = K_0 / (c\rho) \) and

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \]

in Cartesian coordinates.
The **heat equation** in higher dimensions is:

\[ c\rho \frac{\partial u}{\partial t} = \nabla \cdot (K_0 \nabla u) + Q. \]

If the Fourier coefficient is constant, \( K_0 \), then the **Steady-State** problem can be written:

\[ \nabla^2 u = -\frac{Q}{K_0}, \]

which is **Poisson’s equation**

Furthermore, if there are no sources or sinks \((Q \equiv 0)\), then we obtain **Laplace’s equation**

\[ \nabla^2 u = 0. \]
Laplacian in 2D

In Cartesian coordinates, the Laplacian in 2D is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Recall that in polar coordinates

$$x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta).$$

By using the \textbf{chain rule} and the \textbf{dot product}, we find:

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$
Laplacian in 3D

In cylindrical coordinates

\[ x = r \cos(\theta), \quad y = r \sin(\theta), \quad \text{and} \quad z = z, \]

so

\[ \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}. \]

In spherical coordinates

\[ x = \rho \sin(\phi) \cos(\theta), \quad y = \rho \sin(\phi) \sin(\theta), \quad \text{and} \quad z = \rho \cos(\phi), \]

so it can be shown (HW exercise):

\[ \nabla^2 u = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin(\phi)} \frac{\partial}{\partial \phi} \left( \sin(\phi) \frac{\partial u}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2(\phi)} \frac{\partial^2 u}{\partial \theta^2}. \]