1. Consider the function \( f(x) = \frac{x^3 \ln(x)}{1-x^2} + \frac{2}{x^2} = \frac{x^3 - \ln(x)}{1-x^2} + 2x^{-2} \). Applying the quotient rule to the first part and the power rule to the second, we have:

\[
f'(x) = \left(1 - x^2\right)(3x^2 - \frac{1}{x}) - \left(x^3 - \ln(x)\right)(-2x) - 2 \cdot 2x^{-3}
\]

\[
= \frac{(1 - x^2)(3x^2 - \frac{1}{x}) + 2x (x^3 - \ln(x))}{1-x^2} - 4x^{-3}
\]

2. Consider the function \( f(x) = \frac{x^2 e^{-x}}{3x+1} + xe^{-x} \). Applying the product and quotient rule to this function, we have:

\[
f'(x) = \frac{(3x + 1)(2x + e^{-x}) - (x^2 - e^{-x})(3)}{(3x + 1)^2} + (-xe^{-x} + 1 \cdot e^{-x})
\]

\[
= \frac{(3x + 1)(2x + e^{-x}) - 3(x^2 - e^{-x})}{(3x + 1)^2} + (1 - x)e^{-x}
\]

3. Consider the function \( f(x) = \sqrt{x} - \frac{1}{e^x} = \frac{\sqrt{x}}{e^x} - e^{-3x} \). From our rules of differentiation, we have

\[
f'(x) = \frac{2 + x}{2(2 + x)^2 \sqrt{x}} - \frac{1}{e^x} + 3e^{-3x} = \frac{2 - x}{2(2 + x)^2 \sqrt{x}} + 3e^{-3x}
\]

4. Consider the function \( f(x) = \frac{8e^{-2x}}{12 + \cos(2x)} \). The quotient rule gives the derivative

\[
f'(x) = \frac{(12 + \cos(2x))(8(-2)e^{-2x}) - (8e^{-2x})(-2 \sin(2x))}{(12 + \cos(2x))^2}
\]

\[
= \frac{16(\sin(2x) - \cos(2x) - 12)e^{-2x}}{(12 + \cos(2x))^2}
\]

5. Consider the function \( y(x) = \frac{x^2}{x^2 + 1} \). The quotient rule finds the derivative

\[
y'(x) = \frac{(x + 1)(2x) - (1)x^2}{(x + 1)^2} = \frac{x(x + 2)}{(x + 1)^2}
\]

The x-intercept occurs where \( y = 0 \), which is \( x = 0 \), so the x and y-intercept is the origin, \((0,0)\). There is no horizontal asymptote, because the exponent in the numerator is higher than that in the denominator. The vertical asymptote occurs when the denominator is zero, or \( x + 1 = 0 \) so \( x = -1 \). At the critical points, \( y' = 0 = \frac{x(x + 2)}{(x + 1)^2} \), so \( x(x + 2) = 0 \). Thus, \( x_{1c} = -2 \) and \( x_{2c} = 0 \). The \( y \) values are \( y_{1c} = \frac{-2^2}{2+1} = -4 \) and \( y_{2c} = \frac{0^2}{0+1} = 0 \). It follows that \((-2, -4)\) is a maximum and \((0, 0)\) is a minimum. The graph appears below to the left.
Problem 5

\[ y = x^2 + x + 1 \]

Problem 6

\[ y = e^x x + 1 \]

6. Consider the function \( y(x) = \frac{e^x}{x + 1} \). The quotient rule finds the derivative

\[ y' = \frac{(x + 1)e^x - (1)e^x}{(x + 1)^2} = \frac{xe^x}{(x + 1)^2}. \]

The \( x \)-intercept occurs when \( y = 0 \). Since the exponential function is not zero, there are no \( x \)-intercepts. The \( y \)-intercept occurs when \( x = 0 \) or \( y(0) = \frac{e^0}{0+1} = 1 \), so the \( y \)-intercept is \((0, 1)\). Since the denominator \( x + 1 = 0 \) when \( x = -1 \), this is a vertical asymptote. From the limit below,

\[ \lim_{x \to -\infty} \frac{e^x}{x + 1} = 0, \]

so there is a horizontal asymptote to the left at \( y = 0 \). The critical point satisfies \( y' = 0 \), so \( 0 = xe^x \) or \( x_c = 0 \). Since \( y(x_c) = 1 \), we have \((0, 1)\) is a minimum. The graph appears above on the right.

7. Consider the function \( y(x) = \frac{x^2 - 2x + 2}{x - 1} \). The quotient rule finds the derivative

\[ y' = \frac{(x - 1)(2x - 2) - (1)(x^2 - 2x + 2)}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}. \]

The \( x \)-intercept occurs when \( y = 0 \), but \( x^2 - 2x + 2 = 0 \) has no real solution, so there is no \( x \)-intercept. The \( y \)-intercept occurs when \( x = 0 \), so \( y(0) = -2 \). There is no horizontal asymptote, as the highest exponent in the numerator is larger than that in the denominator. There is a vertical asymptote where the denominator is zero, or \( x = 1 \). The critical points satisfy \( y' = 0 \), so \( x(x - 2) = 0 \). It follows that \( x_{1c} = 0 \) and \( y(x_{1c}) = -2 \), which gives a maximum at \((0, -2)\). Similarly, \( x_{2c} = 2 \) and \( y(x_{2c}) = 2 \), which gives a minimum at \((2, 2)\). The graph is shown below on the left.
8. Consider the function \( y = \frac{1}{\sin(3x)} \). Applying the quotient rule,
\[
y'(x) = \frac{-3\cos(3x)}{\sin^2(3x)}.
\]
The period is given by \( T = \frac{2\pi}{3} \). Critical points occur when \( y' = 0 \) or \( \cos(3x) = 0 \), which occurs when \( 3x = \frac{\pi}{2} \) or \( \frac{3\pi}{2} \). Then \( y\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1 \) and \( y\left(\frac{\pi}{2}\right) = \frac{1}{\sin\left(\frac{3\pi}{2}\right)} = -1 \). \( \left(\frac{\pi}{6}, 1\right) \) is a minimum, and \( \left(\frac{\pi}{2}, -1\right) \) is a maximum. The vertical asymptotes occur where \( \sin(3x) = 0 \) or \( x = 0, \frac{\pi}{2}, \frac{2\pi}{3}, \text{ and } \pi \). The curve of the function appears above on the right.

9. a. Consider the function \( y(p) = \frac{p^4}{0.0625 + p^4} \). The derivative satisfies
\[
y'(p) = \frac{(0.0625 + p^4)(4p^3) - (p^4)(4p^3)}{(0.0625 + p^4)^2} = \frac{0.25p^3}{(0.0625 + p^4)^2}.
\]
The second derivative can be found with a second application of the rule:
\[
y''(p) = \frac{0.25p^3}{(0.0625 + p^4)^2} = \frac{0.25p^3(0.75p^2 - 0.25p^3)(0.0625 + p^4)}{(0.0625 + p^4)^4} = \frac{0.25p^2(0.1875 - 5p^4)}{(0.0625 + p^4)^3}.
\]
When \( y''(p) = 0 \), either \( p = 0 \) or \( 5p^4 = 0.1875 \), giving \( p_i \approx 0.44056 \) for \( p > 0 \). At this point, \( y(p_i) = \frac{0.0375}{0.0625 + 0.0375} = 0.375 \), and \( y'(p_i) = \frac{0.25(0.44056)^3}{(0.0625 + 0.0375)^3} = 2.13041 \).

b. There is a \( y \)-intercept when \( p = 0 \), and \( y(0) = 0 \), so there is only one intercept at \( (0,0) \). There is a horizontal asymptote at \( y = \frac{p^2}{p^4} = 1 \). There is no vertical asymptote. The graph of \( y(p) \) is shown below on the left. The graph of \( y'(p) \) is shown below on the right.

c. The function reaches a 90% saturation when \( y(p) = 0.9 = \frac{p^4}{0.0625 + p^4} \). Thus,
\[ 0.9(0.0625 + p^4) = p^4 \quad \text{or} \quad p^4 = 0.9 \cdot 0.625 = 0.5625 \quad \text{or} \quad p = \sqrt[4]{0.5625} \approx 0.866025. \]
This curve is similar in shape to the \( O_2 \) dissociation curve, but the point of inflection occurs at \( p = 21.2 \text{ torr} \), which is about 50 times higher than the point of inflection for the \( CO \) dissociation curve, which implies that hemoglobin binds CO much more strongly than \( O_2 \).
10. a. The specific function is \( R(L) = \frac{10L^2}{1+L^2} \). Differentiating this rate function using the quotient rule gives

\[
R'(L) = \frac{(1+L^2)10(2L) - (10L^2)(2L)}{(1+L^2)^2} = \frac{20L}{(1+L^2)^2} = \frac{20L}{(1+2L^2+L^4)}.
\]

The second derivative satisfies:

\[
R''(L) = \frac{(1+2L^2+L^4)(20) - (20L)(2\cdot2L+4L^3)}{(1+L^2)^4} = \frac{20(1-3L^2)}{(1+L^2)^3}.
\]

At the point of inflection, the second derivative is 0, when the numerator of the above expression is zero. Thus, \( 1 - 3L^2 = 0 \) or \( L = \frac{1}{\sqrt{3}} \approx 0.5774 \). (Note we take only the positive square root because of the domain.) \( R\left(\frac{1}{\sqrt{3}}\right) = \frac{10\left(\frac{1}{\sqrt{3}}\right)}{1+(\frac{1}{3})} = \frac{10}{4} = 2.5 \). Thus, there is a point of inflection at \((0.5774, 2.5)\). At this point, \( R'(0.5774) = \frac{20L}{(1+0.5774^2)^2} = \frac{20.05774}{(1+0.5774^2)^2} = 6.495 \).

b. At the \( R \)-intercept, \( L = 0 \), so there is an intercept at \((0,0)\). Since the highest power of \( L \) in the numerator and denominator is two, there is a horizontal asymptote at \( y = \frac{10}{1} = 10 \), using the leading coefficients of the highest powers. There is no vertical asymptote. The graph of \( R(L) \) is shown below to the left.

c. The only intercept for the derivative is \((0,0)\). The power of the denominator is greater than the power of \( L \) in the numerator, so the horizontal asymptote is \( R'(L) = 0 \). A sketch of \( R'(L) \) is shown below to the right. Since the second derivative is zero at \( L = 0.5774 \), there is a maximum for \( R'(L) \) at \((0.5774, 6.495)\). Clearly, the \( L \)-value of the maximum matches the \( L \)-value for the point of inflection.

11. a. We consider the Beverton-Holt (Hassell’s) model, which is given by \( H(P) = \frac{5P}{1+0.004P} \). Differentiating using the quotient rule,

\[
H'(P) = \frac{5(1+0.004P) - P(0.004)}{(1+0.004P)^2} = \frac{5}{(1+0.004P)^2} = \frac{5}{1+0.008P+0.000016P^2}.
\]
The second derivative is given by

\[ H''(P) = \frac{0 - 5(0.008 + 2 \cdot 0.000016P)}{(1 + 0.004P)^4} = \frac{-0.04}{(1 + 0.004P)^3}. \]

Note that \( H''(P) \) is negative for \( P \geq 0 \).

b. The \( P \) and \( H \)-intercept occurs when \( P = 0 \) and \( H(0) = 0 \). Since the leading powers of the numerator and denominator are the same (one), there is a horizontal asymptote at \( H = \frac{5}{0.004} = 1250 \). There is no vertical asymptote (for \( P \geq 0 \)). A graph of \( H(P) \) is shown below on the left.

c. Equilibria are found by solving

\[ P_e = \frac{5P_e}{1 + 0.004P_e}. \]

This gives \( P_e(1 + 0.004P_e) = 5P_e \), so there is the extinction equilibrium, \( P_e = 0 \), or \( 1 + 0.004P_e = 5 \). This last equation gives the carrying capacity equilibrium, \( P_e = 1000 \).

12. a. Consider the logistic growth model, \( Y(t) = \frac{1000}{1 + 19e^{-0.1t}} \). The derivative comes from the quotient rule,

\[ Y'(t) = \frac{1000(0 + 1.9e^{-0.1t})}{(1 + 19e^{-0.1t})^2} = \frac{1900e^{-0.1t}}{(1 + 19e^{-0.1t})^2} = \frac{1900e^{-0.1t}}{(1 + 38e^{-0.1t} + 361e^{-0.2t})}. \]

The second derivative is again found by the quotient rule

\[ Y''(t) = \frac{-190(1 + 38e^{-0.1t} + 361e^{-0.2t})e^{-0.1t} - 1900e^{-0.1t}(-3.8e^{-0.1t} - 72.2e^{-0.2t})}{(1 + 38e^{-0.1t} + 361e^{-0.2t})^4} \]

\[ = \frac{-190e^{-0.1t}(1 + 19e^{-0.1t})(-1 + 19e^{-0.1t})}{(1 + 19e^{-0.1t} - 1)(1 + 19e^{-0.1t})^3} \]

\[ = \frac{190e^{-0.1t}(19e^{-0.1t} - 1)}{(1 + 19e^{-0.1t})^4}. \]

The second derivative is zero when \( 19e^{-0.1t} - 1 = 0 \) or \( e^{0.1t} = 19 \). or \( t = 10\ln(19) \approx 29.44 \). Then \( Y(29.44) = \frac{1000}{1 + 19e^{-2.944}} \approx 500 \). Thus, there is a point of inflection at (29.44, 500).

b. There is a \( Y \)-intercept at \( t = 0 \), when \( Y(0) = \frac{1000}{1 + 19} = 50 \). The only intercept is (0, 50). Since \( \lim_{t \to \infty} e^{-0.1t} \to 0 \),

\[ \lim_{t \to \infty} Y(t) \to 1000, \]

\[ \lim_{t \to \infty} Y(t) \to 1000, \]
which gives a horizontal asymptote of $Y = 1000$. A sketch of $Y(t)$ is above to the right.

c. Since the population starts at 50, it doubles when it reaches 100. Solving $Y(t) = \frac{1000}{1 + 19e^{-0.1t}} = 100$ gives $1 + 19e^{-0.1t} = 10$, so $e^{-0.1t} = \frac{9}{19}$. Thus, this population doubles when $t = 10 \ln(\frac{19}{9}) \approx 7.47$ hr.

d. The Malthusian growth model doubles when it reaches 100. Solving $100 = 50e^{0.1t}$ gives $e^{0.1t} = 2$ or $t = 10 \ln(2)$. Thus, the doubling time for the Malthusian growth model is $t \approx 6.93$ hr, which is less than for the logistic growth model.