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Introduction

The previous section showed the definition of a derivative
Using the definition of the derivative is not an efficient way to find derivatives
Develop some rules for differentiation
Basic power rule for differentiation
Additive and scalar multiplication rules
Applications to polynomials
Pulse and Weight

- Obtained data from Altman and Dittmer for the pulse and weight of mammals
- The pulse, $P$, as a function of the weight, $w$, are approximated by the relationship
  \[ P = 200w^{-1/4} \]
- The pulse is in beats/min, and the weight is in kilograms
Pulse vs. Weight of Mammals

Pulse (beats/min)

Weight (kg)
Pulse and Weight

- The graph shows an initial steep decrease in the pulse as weight increases.

- Can one quantify how fast the pulse rate changes as a function of weight?

- For small animals the pulse rate changes more rapidly than for large animals.

- The derivative of this allometric or power law model provides more details on the rate of change in pulse rate as a function of weight.
Notation for the Derivative

- There are several standard notations for the derivative
- For the function $f(x)$, the notation that Leibnitz used was
  \[ \frac{df(x)}{dx} \]
- The Newtonian notation for the derivative is written as follows:
  \[ f'(x) \]
- We will use these notations interchangeably
Power Rule

The power rule for differentiation is given by the formula

\[ \frac{d(x^n)}{dx} = nx^{n-1}, \quad \text{for} \quad n \neq 0 \]
Examples of the Power Rule

**Examples:** Differentiate the following functions:

If \( f(x) = x^5 \)

The **derivative** is

\[
    f'(x) = 5x^4
\]

If \( f(x) = x^{-3} \)

The **derivative** is

\[
    f'(x) = -3x^{-4}
\]

If \( f(x) = x^{1/3} \)

The **derivative** is

\[
    f'(x) = \frac{1}{3}x^{-2/3}
\]
Examples of the Power Rule

**Examples:** Differentiate the following functions:

If \( f(x) = \frac{1}{x^4} \), then \( f(x) = x^{-4} \)

The derivative is

\[
f'(x) = -4x^{-5}
\]

If \( f(x) = \frac{1}{\sqrt{x}} \), then \( f(x) = x^{-1/2} \)

The derivative is

\[
f'(x) = -\frac{1}{2}x^{-3/2}
\]

If \( f(x) = 3 \)

Since \( n = 0 \), the power rule does not apply

However, we know the derivative of a constant is

\[
f'(x) = 0
\]
Scalar Multiplication Rule

Assume that \( k \) is a constant and \( f(x) \) is a differentiable function, then

\[
\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx} f(x)
\]

**Example:** Let \( f(x) = 12x^3 \)

The derivative of \( f(x) \) satisfies

\[
f'(x) = \frac{d}{dx}(12x^3) = 12 \frac{d}{dx}(x^3) = 36x^2
\]
Additive Rule

Assume that \( f(x) \) and \( g(x) \) are differentiable functions, then

\[
\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))
\]

**Example:** Let \( f(x) = 2x^{1/2} + x^4 \)

The derivative of \( f(x) \) satisfies

\[
f'(x) = \frac{d}{dx}(2x^{1/2}) + \frac{d}{dx}(x^4) = x^{-1/2} + 4x^3
\]
Differentiation of Polynomials

Consider the polynomial

\[ f(x) = x^4 + 3x^3 - 8x^2 + 10x - 7 \]

From our rules above, the derivative is

\[ f'(x) = 4x^3 + 9x^2 - 16x + 10 \]

**Example:** Other additive powers are handled similarly

\[ f(x) = x^2 + \frac{3}{x^2} - 8\sqrt{x} + 13 = x^2 + 3x^{-2} - 8x^{1/2} + 13 \]

From our rules above, the derivative is

\[ f'(x) = 2x - 6x^{-3} - 4x^{-1/2} \]
Height of Ball

Suppose a ball is thrown vertically with an initial velocity of $v_0$ and an initial height $h(0) = 0$.

Assume the only acceleration is due to gravity, $g$ and air resistance ignored.

The equation for the height satisfies:

$$h(t) = v_0 t - \frac{gt^2}{2}$$
Velocity of Ball

With the equation for the height

\[ h(t) = v_0 t - \frac{gt^2}{2} \]

The velocity is the derivative of \( h(t) \)

\[ v(t) = h'(t) = v_0 - gt \]

- This uses our 3 rules of differentiation to date
  - The additive property of derivatives allows consideration of each of the terms in the height function separately
  - Each term has a scalar multiple
  - Power rule can be applied to the \( t \) and \( t^2 \) terms
Example: A ball, thrown vertically from a platform without air resistance, satisfies the equation

\[ h(t) = 80 + 64t - 16t^2 \]

- Sketch a graph of the height of the ball, \( h(t) \), as a function of time, \( t \)
- Find the maximum height of the ball and determine when the ball hits the ground
- Give an expression for the velocity, \( v(t) \), as a function of time, \( t \)
- Find the velocity at the times \( t = 0 \), \( t = 1 \), and \( t = 2 \)
- What is the velocity of the ball just before it hits the ground?
Height of the ball

- Factoring \( h(t) = -16(t + 1)(t - 5) \), so the ball hits the ground at \( t = 5 \)
- The vertex of the parabola occurs at \( t = 2 \) with \( h(2) = 144 \text{ ft} \)
- The \( h \)-intercept is \( h(0) = 80 \text{ ft} \)
Velocity of a Ball

Since the height is given by

\[ h(t) = 80 + 64t - 16t^2 \]

so the velocity is

\[ v(t) = h'(t) = 64 - 32t \]

- It follows that

\[ v(0) = 64 \text{ ft/sec}, \quad v(1) = 32 \text{ ft/sec}, \quad v(2) = 0 \text{ ft/sec} \]

- The velocity at the maximum is \( v(2) = 0 \text{ ft/sec} \)

- The ball hits the ground with velocity \( v(5) = -96 \text{ ft/sec} \)
Linear Approximation

Recall that the **tangent line** gives a **linear approximation** of a function near the point of tangency.

- The **derivative** give the slope of this tangent line.
- A point on the curve gives the point of tangency.

This provides easy approximations of a function near a given point.

This technique is often used in **Error Analysis**.
Pulse and Weight Example

The model on pulse rate is,

\[ P = 200w^{-0.25} \]

The power law of differentiation gives

\[ \frac{dP}{dt} = -50w^{-5/4} \]

The negative sign shows the decrease in the pulse rate with increasing weight.
Example for Linear Approximation: Suppose we want to approximate the pulse of a 17 kg animal using our model

\[ P = 200w^{-0.25} \]

- An animal at 16 kg by the allometric model would have a pulse of about 100 (since \( P(16) = 200(16)^{-1/4} = 100 \))
- The power law of differentiation gives

\[ \frac{dP}{dt} = -50w^{-5/4} \]

- The derivative at \( w = 16 \) is

\[ P'(16) = -50(16)^{-5/4} = -\frac{50}{32} \approx -1.56 \]
Example for Linear Approximation (cont):

- The tangent line approximation, $P_L(w)$, near $w = 16$ is

  $$P_L(w) = -\frac{50}{32}(w - 16) + 100$$

- It follows that a 17 kg animal should have a pulse near

  $$P_L(17) = -\frac{50}{32}(1) + 100 \approx 98.44 \text{ beats/min}$$

- Note that the Allometric model gives

  $$P(17) = 200(17)^{-1/4} = 98.50 \text{ beats/min}$$
Biodiversity and Area

- Data are collected on the number of species of herpatofauna, $N$, on Caribbean islands with area, $A$
- An allometric model approximates this biodiversity

$$N = 3A^{1/3}$$

- A model of this sort is important for obtaining information about biodiversity
Biodiversity and Area

Biodiversity vs. Island Area

Area (mi$^2$) vs. Herpetofauna Species (N)

Herpetofauna Species (N) vs. Area (mi$^2$)

0 10 20 30 40 50 60 70 80 90 100

0 1 2 3 4 5 $\times 10^4$

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Biodiversity and Area

Can we use this model to determine the rate of change of numbers of species with respect to a given increase in area?

Again the derivative is used to help quantify the rate of change of the dependent variable, $N$, with respect to the independent variable, $A$. 
Biodiversity Example

The model on diversity is,

\[ N = 3 A^{1/3} \]

The power law of differentiation gives

\[ \frac{dN}{dt} = A^{-2/3} \]

- This shows the rate of change of numbers of species with respect to the island area is increasing as the derivative is positive.
- The increase gets smaller with increasing island area, since the area has the power \(-2/3\), which puts the area in the denominator of this expression for the derivative.
Example for Linear Approximation: Suppose we want to approximate the number of species on an island with 950 sq mi

\[ N = 3 \, A^{1/3} \]

- An island with 1000 sq mi by the allometric model would have approximately 30 species (since \( N(1000) = 3(1000)^{1/3} = 30 \))
- The power law of differentiation gives

\[ \frac{dN}{dt} = A^{-2/3} \]

- The derivative at \( A = 1000 \) is

\[ N'(1000) = (1000)^{-2/3} = 0.01 \]
Example for Linear Approximation (cont):

- The tangent line approximation, $N_L(A)$, near $A = 1000$ is

$$N_L(A) = 0.01(A - 1000) + 30$$

- It follows that an island with an area of 950 sq mi should have approximately

$$N_L(950) = 0.01(950 - 1000) + 30 = 29.5 \text{ species}$$

- Note that the Allometric model gives

$$N(950) = 3(950)^{1/3} = 29.49 \text{ species}$$
Logistic Growth Model

A common model in population biology is the **logistic growth model** given by

\[ P_{n+1} = P_n + rP_n \left(1 - \frac{P_n}{M}\right) \]

- Studied the **discrete Malthusian growth model**
  - The growth of the population is proportional to the existing population, \( P_{n+1} = P_n + rP_n \)
  - Malthusian growth model is based on unlimited resources

- As the population increases, the growth rate of most organisms slows
  - Crowding (lack of space to reproduce)
  - Lack of resources (limited food supply)
  - Build up of waste (toxicity)
Logistic Growth Model

The **logistic growth model**

\[ P_{n+1} = P_n + rP_n \left( 1 - \frac{P_n}{M} \right) = P_n + G(P_n) \]

- First part is same as Malthusian growth model
- Quadratic term reflects slowing of growth with increasing population, **growth function**, \( G(P_n) \)
- Nonlinear model, which can have complicated behavior (observe later in Lab)
- For low \( r \) values, model gives classic **S-shaped curve**
- Population reaches an equilibrium, the **carrying capacity**
Example of Logistic Growth Function: Suppose a culture of yeast has the growth function

\[ G(P) = rP \left( 1 - \frac{P}{M} \right) \]

where \( P \) is the density of yeast (\( \times 1000/\text{cc} \))

- Suppose experimental measurements find the growth parameters
  - The Malthusian growth rate \( r = 0.1 \)
  - The parameter \( M = 500 \)
The population is at equilibrium when the growth function is zero

\[ G(P) = 0.1 \cdot P \left(1 - \frac{P}{500}\right) = 0 \]

- This quadratic growth function is in factored form, so equilibria are easily found
  - The extinction equilibrium, \( P_e = 0 \)
  - The carrying capacity, \( P_e = M = 500 \)
The maximum growth occurs at the vertex of the growth function.

Also, the maximum is when the slope of the tangent line is zero or the derivative is zero.

Since

\[ G(P) = 0.1P - \frac{0.1P^2}{500} \]

the derivative is

\[ G'(P) = 0.1 - \frac{0.2P}{500} \]

\[ G'(P) = 0 \text{ when } P = 250 \]
This model gives **equilibria** at \( P_e = 0 \) and \( P_e = 500 \ (\times 1000) \) yeast/cc

Maximum population growth occurs at \( P_v = 250 \ (\times 1000) \) yeast/cc

Since \( G(250) = 12.5 \), when the **density of yeast** is 250 \((\times 1000)\) yeast/cc, the maximum production is 12.5 \((\times 1000)\) yeast/cc/hr
Suppose the population begins with $P_0 = 50 \times 1000$ yeast/cc

Below shows the simulation of

$$P_{n+1} = P_n + 0.1 P_n \left( 1 - \frac{P_n}{500} \right)$$

for $0 \leq n \leq 80$ hr

Simulation shows the population approaching the **carrying capacity** of 500 and the maximum growth near $n = 25$