7.7.1 (15 pts) Consider the vibrating membrane satisfying

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),$$

with BC $u(a,\theta,t) = 0$ and ICs $u(r,\theta,0) = 0$ and $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin(3\theta)$. We apply separation of variables: $u(r,\theta,t) = \phi(r)g(\theta)h(t)$, which gives

$$\phi g h'' = c^2\left(\frac{gh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{\phi h}{r^2}\frac{d^2 g}{d\theta^2}\right),$$

which is equivalent to

$$\frac{h''}{c^2 h} = \left(\frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{1}{r^2 g}\frac{d^2 g}{d\theta^2}\right) = -\lambda.$$

This leads to the t-equation: $h'' + \lambda c^2 h = 0$, which has the solution:

$$h(t) = c_1\cos(c\sqrt{\lambda}t) + c_2\sin(c\sqrt{\lambda}t).$$

A second separation of variables gives two Sturm-Liouville problems:

$$\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + r^2\lambda = -\frac{g''}{g} = \mu.$$

The first Sturm-Liouville problem is:

$$g'' + \mu g = 0, \quad \text{with} \quad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

This is a standard eigenvalue problem with periodic boundary conditions, which we have solved before. This has eigenvalue $\mu_0 = 0$ with eigenfunction $g_0(\theta) = 1$ and eigenvalues $\mu_m = m^2$ with eigenfunctions $g_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta)$.

The second Sturm-Liouville problem is:

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + r^2\sqrt{\lambda}\phi - m^2\phi = 0,$$

which is Bessel's equation of order m. This has the general solution:

$$\phi(r) = c_1 J_m(\sqrt{\lambda}r) + c_2 Y_m(\sqrt{\lambda}r).$$

The boundedness condition at the origin requires $c_2 = 0$, so the eigenvalue problem gives:

$$\phi_{mn}(r) = c_1 J_m\left(\sqrt{\lambda_{mn}}r\right),$$

where $J_m\left(\sqrt{\lambda_{mn}}a\right) = 0$ with $\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2 > 0$ with z_{mn} being the n^{th} zero of m^{th} order Bessel function ($J_m(z_{mn}) = 0$).

The Superposition principle gives:

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=1}^{\infty} J_0\left(\sqrt{\lambda_{0n}}r\right)\left(A_{0n}\cos\left(c\sqrt{\lambda_{0n}}t\right)+B_{0n}\sin\left(c\sqrt{\lambda_{0n}}t\right)\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(m\theta)J_m\left(\sqrt{\lambda_{mn}}r\right)\left(A_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)+B_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(m\theta)J_m\left(\sqrt{\lambda_{mn}}r\right)\left(C_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)+D_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right). \end{aligned}$$

The initial displacement is zero ($u(r,\theta,0)=0$), which gives $A_{mn}=0$ for $m=0,1,2...$ and $n=1,2,...$ and $C_{mn}=0$ for $m=1,2...$ and $n=1,2,....$ This reduces our solution to

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=1}^{\infty} B_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right)\sin\left(c\sqrt{\lambda_{0n}}t\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m\left(\sqrt{\lambda_{mn}}r\right)\left(B_{mn}\cos(m\theta)+D_{mn}\sin(m\theta)\right)\sin\left(c\sqrt{\lambda_{mn}}t\right). \end{aligned}$$

The initial velocity gives $\frac{\partial u}{\partial t}(r,\theta,0)=\alpha(r)\sin(3\theta)$. From the solution above we have

$$\begin{aligned} \frac{\partial u}{\partial t}(r,\theta,t) &= \sum_{n=1}^{\infty} c\sqrt{\lambda_{0n}}B_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right)\cos\left(c\sqrt{\lambda_{0n}}t\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}J_m\left(\sqrt{\lambda_{mn}}r\right)\left(B_{mn}\cos(m\theta)+D_{mn}\sin(m\theta)\right)\cos\left(c\sqrt{\lambda_{mn}}t\right), \end{aligned}$$

which when evaluated at $t=0$ gives:

$$\begin{aligned} \frac{\partial u}{\partial t}(r,\theta,0) &= \sum_{n=1}^{\infty} c\sqrt{\lambda_{0n}}B_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}J_m\left(\sqrt{\lambda_{mn}}r\right)\left(B_{mn}\cos(m\theta)+D_{mn}\sin(m\theta)\right) \\ &= \alpha(r)\sin(3\theta). \end{aligned}$$

Orthogonality of the eigenfunctions in θ implies that $B_{mn}=0$ for $m=0,1,2,...$ and $n=1,2,...,$ and $D_{mn}=0$ for all $m\neq 3$ and $n=1,2,...$ The only nonzero coefficients are:

$$D_{3n}=\frac{\int_0^a \alpha(r)J_3\left(\sqrt{\lambda_{3n}}r\right)r\,dr}{c\sqrt{\lambda_{3n}}\int_0^a J_3^2\left(\sqrt{\lambda_{3n}}r\right)r\,dr}.$$

With these Fourier coefficients, it follows that the solution satisfies:

$$u(r,\theta,t)=\sum_{n=1}^{\infty} D_{3n}J_3\left(\sqrt{\lambda_{3n}}r\right)\sin(3\theta)\sin\left(c\sqrt{\lambda_{3n}}t\right).$$

7.7.2.a. (15 pts) Consider the circular membrane:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),$$

with BC $\frac{\partial u}{\partial r}(a,\theta,t) = 0$ and ICs $u(r,\theta,0) = 0$ and $\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r)\cos(5\theta)$. This begins just like Problem 7.7.1. We apply separation of variables: $u(r,\theta,t) = \phi(r)g(\theta)h(t)$, which gives

$$\phi g h'' = c^2\left(\frac{gh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{\phi h}{r^2}\frac{d^2 g}{d\theta^2}\right),$$

which is equivalent to

$$\frac{h''}{c^2 h} = \left(\frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{1}{r^2 g}\frac{d^2 g}{d\theta^2}\right) = -\lambda.$$

This leads to the t-equation: $h'' + \lambda c^2 h = 0$, which has the solution:

$$h(t) = \begin{cases} c_1 t + c_2, & \lambda = 0, \\ c_1\cos(c\sqrt{\lambda}t) + c_2\sin(c\sqrt{\lambda}t), & \lambda > 0. \end{cases}$$

A second separation of variables gives two Sturm-Liouville problems:

$$\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + r^2\lambda = -\frac{g''}{g} = \mu.$$

The first Sturm-Liouville problem is:

$$g'' + \mu g = 0, \quad \text{with} \quad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

This is a standard eigenvalue problem with periodic boundary conditions, which we have solved before. This has eigenvalue $\mu_0 = 0$ with eigenfunction $g_0(\theta) = 1$ and eigenvalues $\mu_m = m^2$ with eigenfunctions $g_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta)$.

The second Sturm-Liouville problem is:

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + r^2\sqrt{\lambda}\phi - m^2\phi = 0,$$

which is Bessel's equation of order m. This has the general solution:

$$\phi(r) = \begin{cases} d_1 + d_2\ln(r), & \lambda = 0 \quad \text{and} \quad m = 0, \\ d_1 J_m(\sqrt{\lambda}r) + d_2 Y_m(\sqrt{\lambda}r), & \lambda > 0 \quad \text{and} \quad m = 0, 1, 2, ... \end{cases}$$

The boundedness condition at the origin requires $d_2 = 0$, so

$$\phi'(r) = \begin{cases} 0, & \lambda = 0 \quad \text{and} \quad m = 0, \\ d_1\sqrt{\lambda}J_m'(\sqrt{\lambda}r), & \lambda > 0 \quad \text{and} \quad m = 0, 1, 2, ... \end{cases}$$

The BVP requires $\phi'(a) = 0$, so one eigenfunction is

$$\phi_{00}(r) = 1, \qquad \text{with} \qquad \lambda = 0 \quad \text{and} \quad m = 0.$$

The other eigenfunctions, where $\lambda > 0$ and $m = 0, 1, 2, ...$, are

$$\phi_{mn}(r) = J_m\left(\sqrt{\lambda_{mn}}r\right),$$

where $J_m'\left(\sqrt{\lambda_{mn}}a\right)=0$ with $\lambda_{mn}=\left(\frac{z_{mn}}{a}\right)^2$ with z_{mn} being the n^{th} zero of the derivative of m^{th} order Bessel function ($J_m'(z_{mn})=0$).

The Superposition principle gives:

$$\begin{aligned} u(r,\theta,t) &= A_1 t + A_0 + \sum_{n=1}^{\infty} J_0\left(\sqrt{\lambda_{0n}}r\right)\left(B_{0n}\cos\left(c\sqrt{\lambda_{0n}}t\right)+C_{0n}\sin\left(c\sqrt{\lambda_{0n}}t\right)\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(m\theta)J_m\left(\sqrt{\lambda_{mn}}r\right)\left(B_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)+C_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(m\theta)J_m\left(\sqrt{\lambda_{mn}}r\right)\left(D_{mn}\cos\left(c\sqrt{\lambda_{mn}}t\right)+E_{mn}\sin\left(c\sqrt{\lambda_{mn}}t\right)\right). \end{aligned}$$

The initial displacement is zero ($u(r,\theta,0)=0$), which gives $A_0=0$, $B_{mn}=0$ for $m=0,1,2...$ and $n=1,2,...$ and $D_{mn}=0$ for $m=1,2...$ and $n=1,2,....$ This reduces our solution to

$$\begin{aligned} u(r,\theta,t) &= A_1 t + \sum_{n=1}^{\infty} C_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right)\sin\left(c\sqrt{\lambda_{0n}}t\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m\left(\sqrt{\lambda_{mn}}r\right)\left(C_{mn}\cos(m\theta)+E_{mn}\sin(m\theta)\right)\sin\left(c\sqrt{\lambda_{mn}}t\right). \end{aligned}$$

The initial velocity gives $\frac{\partial u}{\partial t}(r,\theta,0)=\beta(r)\cos(5\theta)$. From the solution above we have

$$\begin{aligned} \frac{\partial u}{\partial t}(r,\theta,t) &= A_1 + \sum_{n=1}^{\infty} c\sqrt{\lambda_{0n}}C_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right)\cos\left(c\sqrt{\lambda_{0n}}t\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}J_m\left(\sqrt{\lambda_{mn}}r\right)\left(C_{mn}\cos(m\theta)+E_{mn}\sin(m\theta)\right)\cos\left(c\sqrt{\lambda_{mn}}t\right), \end{aligned}$$

which when evaluated at $t=0$ gives:

$$\begin{aligned} \frac{\partial u}{\partial t}(r,\theta,0) &= A_1 + \sum_{n=1}^{\infty} c\sqrt{\lambda_{0n}}C_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}}J_m\left(\sqrt{\lambda_{mn}}r\right)\left(C_{mn}\cos(m\theta)+E_{mn}\sin(m\theta)\right) \\ &= \beta(r)\cos(5\theta). \end{aligned}$$

Orthogonality of the eigenfunctions in θ implies that $A_1=0$, $C_{mn}=0$ for $m\neq 5$ and $n=1,2,...,$ and $E_{mn}=0$ for $m=1,2,...$ and $n=1,2,...$ The only nonzero coefficients are:

$$C_{5n}=\frac{\int_0^a \beta(r)J_5\left(\sqrt{\lambda_{5n}}r\right)r\,dr}{c\sqrt{\lambda_{5n}}\int_0^a J_5^2\left(\sqrt{\lambda_{5n}}r\right)r\,dr}.$$

With these Fourier coefficients, it follows that the solution satisfies:

$$u(r,\theta,t)=\sum_{n=1}^{\infty} C_{5n}J_5\left(\sqrt{\lambda_{5n}}r\right)\cos(5\theta)\sin\left(c\sqrt{\lambda_{5n}}t\right).$$

7.9.1.c. (15 pts) Consider Laplace's equation on a cylinder:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

with BCs $u(r,\theta,0) = 0$, $u(r,\theta,H) = \beta(r)\cos(3\theta)$, and $\frac{\partial u}{\partial r}(a,\theta,z) = 0$ (insulated). There are implicit BCs $u(r,-\pi,z) = u(r,\pi,z)$, $u_\theta(r,-\pi,z) = u_\theta(r,\pi,z)$, and $u(0,\theta,z)$ bounded. We apply separation of variables $u(r,\theta,z) = \phi(r)g(\theta)h(z)$, giving:

$$\frac{gh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{\phi h g''}{r^2} + \phi g h'' = 0, \qquad \text{or} \qquad \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{g''}{r^2 g} = -\frac{h''}{h} = -\lambda.$$

This gives the z equation: $h'' - \lambda h = 0$, which has the solution:

$$h(z) = \begin{cases} c_1 z + c_2, & \lambda = 0, \\ c_1\cosh(\sqrt{\lambda}z) + c_2\sinh(\sqrt{\lambda}z), & \lambda > 0. \end{cases}$$

A second separation gives two Sturm-Liouville problems:

$$\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r^2 = -\frac{g''}{g} = \mu.$$

The first Sturm-Liouville problem is:

$$g'' + \mu g = 0, \quad \text{with} \quad g(-\pi) = g(\pi) \quad \text{and} \quad g'(-\pi) = g'(\pi).$$

This is a standard eigenvalue problem with periodic boundary conditions, which we have solved before. This has eigenvalue $\mu_0 = 0$ with eigenfunction $g_0(\theta) = 1$ and eigenvalues $\mu_m = m^2$ with eigenfunctions $g_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta)$.

The second Sturm-Liouville problem is:

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + r^2\sqrt{\lambda}\phi - m^2\phi = 0,$$

which is Bessel's equation of order m. This has the general solution:

$$\phi(r) = \begin{cases} d_1 + d_2\ln(r), & \lambda = 0 \quad \text{and} \quad m = 0, \\ d_1 J_m(\sqrt{\lambda}r) + d_2 Y_m(\sqrt{\lambda}r), & \lambda > 0 \quad \text{and} \quad m = 0, 1, 2, ... \end{cases}$$

The boundedness condition at the origin requires $d_2 = 0$, so

$$\phi'(r) = \begin{cases} 0, & \lambda = 0 \quad \text{and} \quad m = 0, \\ d_1\sqrt{\lambda}J_m'(\sqrt{\lambda}r), & \lambda > 0 \quad \text{and} \quad m = 0, 1, 2, ... \end{cases}$$

The BVP requires $\phi'(a) = 0$, so one eigenfunction is

$$\phi_{00}(r) = 1, \qquad \text{with} \qquad \lambda = 0 \quad \text{and} \quad m = 0.$$

The other eigenfunctions, where $\lambda > 0$ and $m = 0, 1, 2, ...$, are

$$\phi_{mn}(r) = J_m\left(\sqrt{\lambda_{mn}}r\right),$$

where $J_m'\left(\sqrt{\lambda_{mn}}a\right) = 0$ with $\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$ with z_{mn} being the n^{th} zero of the derivative of m^{th} order Bessel function ($J_m'(z_{mn}) = 0$).

The Superposition principle gives:

$$
\begin{aligned}
u(r,\theta,z) \;=\;& A_1 z + A_0 + \sum_{n=1}^{\infty} J_0\left(\sqrt{\lambda_{0n}}r\right)\left(B_{0n}\cosh\left(\sqrt{\lambda_{0n}}z\right) + C_{0n}\sinh\left(\sqrt{\lambda_{0n}}z\right)\right)\\
&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(m\theta)J_m\left(\sqrt{\lambda_{mn}}r\right)\left(B_{mn}\cosh\left(\sqrt{\lambda_{mn}}z\right) + C_{mn}\sinh\left(\sqrt{\lambda_{mn}}z\right)\right)\\
&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin(m\theta)J_m\left(\sqrt{\lambda_{mn}}r\right)\left(D_{mn}\cosh\left(\sqrt{\lambda_{mn}}z\right) + E_{mn}\sinh\left(\sqrt{\lambda_{mn}}z\right)\right).
\end{aligned}
$$

The homogeneous BC on the bottom of the cylinder, $u(r,\theta,0) = 0$, gives $A_0 = 0$, $B_{mn} = 0$ for $m = 0,1,2...$ and $n = 1,2,...$ and $D_{mn} = 0$ for $m = 1,2...$ and $n = 1,2,....$ This reduces our solution to

$$
\begin{aligned}
u(r,\theta,z) \;=\;& A_1 z + \sum_{n=1}^{\infty} C_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right)\sinh\left(\sqrt{\lambda_{0n}}z\right)\\
&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m\left(\sqrt{\lambda_{mn}}r\right)\left(C_{mn}\cos(m\theta) + E_{mn}\sin(m\theta)\right)\sinh\left(\sqrt{\lambda_{mn}}z\right).
\end{aligned}
$$

The BC at the top of the cylinder gives $u(r,\theta,H) = \beta(r)\cos(3\theta)$. From the solution above we have

$$
\begin{aligned}
u(r,\theta,H) \;=\;& A_1 H + \sum_{n=1}^{\infty} C_{0n}J_0\left(\sqrt{\lambda_{0n}}r\right)\sinh\left(\sqrt{\lambda_{0n}}H\right)\\
&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m\left(\sqrt{\lambda_{mn}}r\right)\left(C_{mn}\cos(m\theta) + E_{mn}\sin(m\theta)\right)\sinh\left(\sqrt{\lambda_{mn}}H\right)\\
=\;& \beta(r)\cos(3\theta).
\end{aligned}
$$

Orthogonality of the eigenfunctions in θ implies that $A_1 = 0$, $C_{mn} = 0$ for $m \neq 3$ and $n = 1,2,...$, and $E_{mn} = 0$ for $m = 1,2,...$ and $n = 1,2,...$ The only nonzero coefficients are:

$$
C_{3n} = \frac{\int_0^a \beta(r)J_3\left(\sqrt{\lambda_{3n}}r\right)r\,dr}{\sinh\left(\sqrt{\lambda_{3n}}H\right)\int_0^a J_3^2\left(\sqrt{\lambda_{3n}}r\right)r\,dr}.
$$

With these Fourier coefficients, it follows that the solution satisfies:

$$
u(r,\theta,z) = \sum_{n=1}^{\infty} C_{3n}J_3\left(\sqrt{\lambda_{3n}}r\right)\cos(3\theta)\sinh\left(\sqrt{\lambda_{3mn}}H\right).
$$

7.9.2.b. (15 pts) Consider the semi-circular cylinder:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

with BCs $u(r,\theta,0) = 0$, $\frac{\partial u}{\partial z}(r,\theta,H) = 0$, $u(r,0,z) = 0$, $u(r,\pi,z) = 0$, and $u(a,\theta,z) = \beta(\theta,z)$. There is an implicit BC that $u(0,\theta,z)$ is bounded. We apply separation of variables $u(r,\theta,z) = \phi(r)g(\theta)h(z)$, giving:

$$\frac{gh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{\phi h g''}{r^2} + \phi g h'' = 0 \qquad \text{or} \qquad \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{g''}{r^2 g} = -\frac{h''}{h} = \lambda.$$

This gives the first Sturm-Liouville problem:

$$h'' + \lambda h = 0, \qquad \text{with} \qquad h(0) = 0 \quad \text{and} \quad h'(H) = 0.$$

We have shown that $\lambda \leq 0$ leads only to trivial solutions. The solution is:

$$h(z) = c_1 \cos(\sqrt{\lambda} z) + c_2 \sin(\sqrt{\lambda} z),$$

The BC $h(0) = 0$ implies $c_1 = 0$. The BC $h'(H) = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}H) = 0$, which for nontrivial solutions gives:

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2H} \quad \text{or} \quad \lambda_n = \frac{(2n-1)^2\pi^2}{4H^2}, \quad n = 1, 2, ... \qquad \text{with e.f.} \quad h(t) = \sin\left(\tfrac{(2n-1)\pi z}{2H}\right).$$

A second separation gives one Sturm-Liouville problem and a modified Bessel's equation:

$$\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \lambda r^2 = -\frac{g''}{g} = \mu.$$

The second Sturm-Liouville problem is:

$$g'' + \mu g = 0, \quad \text{with} \quad g(0) = 0 \quad \text{and} \quad g(\pi) = 0.$$

This is a standard Dirichlet eigenvalue problem, which we have solved before. This has eigenvalues $\mu_m = m^2$ with eigenfunctions $g_m(\theta) = \sin(m\theta)$ for $m = 1, 2, ...$

The modified Bessel's equation is:

$$\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \left(-\frac{(2n-1)^2\pi^2}{4H^2}r - \frac{m^2}{r}\right)\phi = 0,$$

which has the general solution:

$$\phi(r) = c_1 I_m\left(\frac{(2n-1)\pi r}{2H}\right) + c_2 K_m\left(\frac{(2n-1)\pi r}{2H}\right).$$

The boundedness condition implies $c_2 = 0$.

The Superposition principle gives:

$$u(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(m\theta)\sin\left(\frac{(2n-1)\pi z}{2H}\right) I_m\left(\frac{(2n-1)\pi r}{2H}\right).$$

The nonhomogeneous BC satisfies:

$$\begin{aligned} u(r,\theta,a) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(m\theta)\sin\left(\frac{(2n-1)\pi z}{2H}\right) I_m\left(\frac{(2n-1)\pi a}{2H}\right) \\ &= \beta(\theta,z). \end{aligned}$$

Using the orthogonality of the eigenfunctions, we obtain the Fourier coefficients:

$$A_{mn} = \frac{4\int_0^{\pi}\int_0^{H}\beta(\theta,z)\sin(m\theta)\sin\left(\frac{(2n-1)\pi z}{2H}\right)dz\,d\theta}{\pi H I_m\left(\frac{(2n-1)\pi a}{2H}\right)}.$$