Homework 7

7.3.1. Consider the heat equation in a two-dimensional rectangular region $0 < x < L, 0 < y < H$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

(d) $u(0, y, t) = 0,\quad \frac{\partial u}{\partial x}(L, y, t) = 0,\quad \frac{\partial u}{\partial y}(x, 0, t) = 0,\quad \frac{\partial u}{\partial y}(x, H, t) = 0$

7.3.2. Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L,\ 0 < y < H,\ 0 < z < W$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

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subject to the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

(a) $u(0, y, z, t) = 0,\quad \frac{\partial u}{\partial y}(x, 0, z, t) = 0,\quad \frac{\partial u}{\partial z}(x, y, 0, t) = 0,$

$u(L, y, z, t) = 0,\quad \frac{\partial u}{\partial y}(x, H, z, t) = 0,\quad u(x, y, W, t) = 0$

7.3.4. Consider the wave equation for a vibrating rectangular membrane $(0 < x < L,\ 0 < y < H)$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial conditions

$$u(x, y, 0) = 0 \text{ and } \frac{\partial u}{\partial t}(x, y, 0) = f(x, y).$$

Solve the initial value problem if

(a) $u(0, y, t) = 0,\quad u(L, y, t) = 0,\quad \frac{\partial u}{\partial y}(x, 0, t) = 0,\quad \frac{\partial u}{\partial y}(x, H, t) = 0$

7.3.5. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - k\frac{\partial u}{\partial t} \quad \text{with } k > 0.$$

(a) Give a *brief* physical interpretation of this equation.

(b) Suppose that $u(x, y, t) = f(x)g(y)h(t)$. What ordinary differential equations are satisfied by f, g, and h?

7.5.1. The vertical displacement of a nonuniform membrane satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

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where c depends on x and y. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

(a) Show that the time variable can be separated by assuming that

$$u(x, y, t) = \phi(x, y)h(t).$$

Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2 \phi + \lambda \sigma(x, y)\phi = 0 \quad \text{with} \quad \phi = 0 \quad \text{on the boundary.} \qquad (7.5.12)$$

What is $\sigma(x, y)$?

7.5.2. See Exercise 7.5.1. Consider the two-dimensional eigenvalue problem given in (7.5.12).

(a) Prove that the eigenfunctions belonging to different eigenvalues are orthogonal (with what weight?).