7.3.1.d. (15pts) Consider the heat equation in a 2-dimensional rectangular region

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad 0 < x < L, \quad 0 < y < H,$$

with IC: $u(x, y, 0) = f(x, y)$ and BCs:

$$u(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0.$$

With separation of variables, take $u(x, y, t) = \phi(x)g(y)h(t)$, then

$$h'\phi g = k(\phi'' gh + g''\phi h), \qquad \text{so} \qquad \frac{h'}{kh} = \frac{\phi''}{\phi} + \frac{g''}{g} = -\lambda.$$

The differential equation in t is:

$$h' = -\lambda k h, \qquad \text{so} \qquad h(t) = ce^{-k\lambda t}.$$

A second separation of variables gives:

$$\frac{\phi''}{\phi} = -\frac{g''}{g} - \lambda = -\mu.$$

The first SL-problem is:

$$\phi'' + \mu\phi = 0, \qquad \text{with} \quad \phi(0) = 0, \quad \phi'(L) = 0.$$

If $\mu = -\alpha^2 < 0$, then $\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$. The BCs give $\phi(0) = c_1 = 0$ and $\phi'(L) = c_2\alpha \cosh(\alpha L) = 0$, so $c_2 = 0$, yielding only the trivial solution.
If $\mu = 0$, then $\phi(x) = c_1 x + c_2$. The BCs give $\phi(0) = c_2 = 0$ and $\phi'(L) = c_1 = 0$, yielding only the trivial solution.
If $\mu = \alpha^2 > 0$, then $\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. The BCs give $\phi(0) = c_1 = 0$ and $\phi'(L) = c_2\alpha \cos(\alpha L) = 0$. For non-trivial solutions, $\alpha_m = \frac{(2m-1)\pi}{2L}$, which gives eigenvalues and eigenfunctions:

$$\mu_m = \frac{(2m-1)^2\pi^2}{4L^2} \qquad \text{and} \qquad \phi_m(x) = \sin\left(\frac{(2m-1)\pi x}{2L}\right), \quad m = 1, 2, 3...$$

The second SL-problem is $g'' = (-\lambda + \mu_m)g = -\nu g$ with $g'(0) = 0$ and $g'(H) = 0$. This is a standard Neumann BC problem, which has eigenvalues and eigenfunctions:

$$\nu_0 = 0 \qquad \text{and} \qquad g_0(y) = 1,$$
$$\nu_n = \frac{n^2\pi^2}{H^2} \qquad \text{and} \qquad g_n(y) = \cos\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, 3...$$

We already showed that $\nu < 0$ gives only trivial solutions. It follows that

$$\lambda_{mn} = \mu_m + \nu_n = \frac{(2m-1)^2\pi^2}{4L^2} + \frac{n^2\pi^2}{H^2}, \qquad m = 1, 2, ..., \quad n = 0, 1, 2, ...$$

Superposition principle gives:

$$\begin{aligned}u(x,y,t) &= \sum_{m=1}^{\infty} A_{m0}\sin\left(\frac{(2m-1)\pi x}{2L}\right)e^{-k\frac{(2m-1)^2\pi^2}{4L^2}t}\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{(2m-1)\pi x}{2L}\right)\cos\left(\frac{n\pi y}{H}\right)e^{-k\left(\frac{(2m-1)^2\pi^2}{4L^2}+\frac{n^2\pi^2}{H^2}\right)t}\end{aligned}$$

The initial condition gives:

$$u(x,y,0)=f(x,y)=\sum_{m=1}^{\infty} A_{m0}\sin\left(\frac{(2m-1)\pi x}{2L}\right)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{n\pi y}{H}\right).$$

Using orthogonality, we obtain the Fourier coefficients:

$$\begin{aligned}A_{m0} &= \frac{\int_0^H\int_0^L f(x,y)\sin\left(\frac{(2m-1)\pi x}{2L}\right)\,dx\,dy}{\int_0^H\int_0^L \sin^2\left(\frac{(2m-1)\pi x}{2L}\right)\,dx\,dy}\\ &= \frac{2}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{(2m-1)\pi x}{2L}\right)\,dx\,dy \text{ for } m=1,2,3...\\ A_{mn} &= \frac{\int_0^H\int_0^L f(x,y)\sin\left(\frac{(2m-1)\pi x}{2L}\right)\cos\left(\frac{n\pi y}{H}\right)dx\,dy}{\int_0^H\int_0^L \sin^2\left(\frac{(2m-1)\pi x}{2L}\right)\cos^2\left(\frac{m\pi y}{H}\right)dx\,dy}\\ &= \frac{4}{HL}\int_0^H\int_0^L f(x,y)\sin\left(\frac{(2m-1)\pi x}{2L}\right)\cos\left(\frac{n\pi y}{H}\right)dx\,dy \text{ for } n=1,2,3...,m=1,2,3\end{aligned}$$

Since

$$\lim_{t\to\infty} e^{-k\frac{(2m-1)^2\pi^2}{4L^2}t}=0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-k\left(\frac{(2m-1)^2\pi^2}{4L^2}+\frac{n^2\pi^2}{H^2}\right)t}=0,$$

it follows that:

$$\lim_{t\to\infty} u(x,y,t)=0.$$

7.3.2.a. (20pts) Consider the heat equation in a 3-dimensional box-shaped region, $0<x<L$, $0<y<H$, and $0<z<W$:

$$\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{d\partial^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right),$$

with IC $u(x,y,z,0)=f(x,y,z)$ and BCs:

$$\begin{array}{ccc} u(0,y,z,t)=0, & u(L,y,z,t)=0, & \frac{\partial u}{\partial y}(x,0,z,t)=0,\\ \frac{\partial u}{\partial y}(x,H,z,t)=0, & \frac{\partial u}{\partial z}(x,y,0,t)=0, & u(x,y,W,t)=0.\end{array}$$

Assuming k is a constant, separation of variables gives $u(x,y,z,t)=\phi(x)g(y)s(z)h(t)$, so

$$h'\phi gs=k(\phi''gsh+g''sh\phi+s''h\phi g), \qquad \text{or} \qquad \frac{h'}{kh}=\frac{\phi''}{\phi}+\frac{g''}{g}+\frac{s''}{s}=-\lambda.$$

The differential equation in t is:

$$h' = -\lambda k h, \qquad \text{so} \qquad h(t) = ce^{-k\lambda t}.$$

A second separation of variables gives:

$$\frac{\phi''}{\phi} = -\frac{g''}{g} - \frac{s''}{s} - \lambda = -\mu.$$

The first SL-problem is:

$$\phi'' + \mu\phi = 0, \qquad \text{with} \quad \phi(0) = 0, \quad \phi(L) = 0.$$

This is a standard SL-problem with Dirichlet BC problem, which has eigenvalues and eigenfunctions:

$$\mu_m = \frac{m^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_m(x) = \sin\left(\frac{m\pi x}{L}\right), \quad m = 1, 2, 3...$$

We already showed that $\mu \leq 0$ gives only trivial solutions.

A third separation of variables gives:

$$\frac{g''}{g} = -\frac{s''}{s} - \lambda + \mu = -\nu.$$

The second SL-problem is $g'' + \nu g = 0$ with $g'(0) = 0$ and $g'(H) = 0$. This is a standard Neumann BC problem, which has eigenvalues and eigenfunctions:

$$\begin{aligned} \nu_0 &= 0 \qquad \text{and} \qquad g_0(y) = 1, \\ \nu_n &= \frac{n^2\pi^2}{H^2} \qquad \text{and} \qquad g_n(y) = \cos\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, 3... \end{aligned}$$

We already showed that $\nu < 0$ gives only trivial solutions.

The remaining ODE is:

$$\frac{s''}{s} = \nu + \mu - \lambda = -\gamma.$$

The third SL-problem is $s'' + \gamma s = 0$ with $s'(0) = 0$ and $s(W) = 0$. If $\gamma = -\alpha^2 < 0$, then $s(z) = c_1 \cosh(\alpha z) + c_2 \sinh(\alpha z)$. The BCs give $s'(0) = \alpha c_2 = 0$ or $c_2 = 0$ and $s(W) = c_1 \cosh(\alpha W) = 0$, so $c_1 = 0$, yielding only the trivial solution.
If $\gamma = 0$, then $s(z) = c_1 z + c_2$. The BCs give $s'(0) = c_1 = 0$ and $s(W) = c_2 = 0$, yielding only the trivial solution.
If $\gamma = \alpha^2 > 0$, then $s(z) = c_1 \cos(\alpha z) + c_2 \sin(\alpha z)$. The BCs give $s'(0) = c_2\alpha = 0$ or $c_2 = 0$ and $s(W) = c_1 \cos(\alpha W) = 0$. For non-trivial solutions, $\alpha_p = \frac{(2p-1)\pi}{2W}$, which gives eigenvalues and eigenfunctions:

$$\gamma_p = \frac{(2p-1)^2\pi^2}{4W^2} \qquad \text{and} \qquad s_p(z) = \cos\left(\frac{(2p-1)\pi z}{2W}\right), \quad p = 1, 2, 3...$$

Combining these results gives:

$$\lambda_{mnp} = \mu_m + \nu_n + \gamma_p = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{H^2} + \frac{(2p-1)^2\pi^2}{4W^2}.$$

By the superposition principle:

$$\begin{aligned} u(x,y,z,t) &= \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} A_{m0p}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right)e^{-k\left(\frac{m^2\pi^2}{L^2}+\frac{(2p-1)^2\pi^2}{4W^2}\right)t} \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{mnp}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right)e^{-k\left(\frac{m^2\pi^2}{L^2}+\frac{n^2\pi^2}{H^2}+\frac{(2p-1)^2\pi^2}{4W^2}\right)t}, \end{aligned}$$

The initial condition gives:

$$\begin{aligned} u(x,y,z,0) &= f(x,y,z) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} A_{m0p}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{mnp}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right) \end{aligned}$$

By using orthogonality we obtain the Fourier coefficients:

$$\begin{aligned} A_{m0p} &= \frac{\int_0^W\int_0^H\int_0^L f(x,y,z)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right)dx\,dy\,dz}{\int_0^W\int_0^H\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)\cos^2\left(\frac{(2p-1)\pi z}{2W}\right)dx\,dy\,dz} \\ &= \frac{4}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right)dx\,dy\,dz \\ &\qquad \text{for } m=1,2,3\ldots\ p=1,2,3\ldots \\ A_{mnp} &= \frac{\int_0^W\int_0^H\int_0^L f(x,y,z)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right)dx\,dy\,dz}{\int_0^W\int_0^H\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)\cos^2\left(\frac{n\pi y}{H}\right)\cos^2\left(\frac{(2p-1)\pi z}{2W}\right)dx\,dy\,dz} \\ &= \frac{8}{LHW}\int_0^W\int_0^H\int_0^L f(x,y,z)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)\cos\left(\frac{(2p-1)\pi z}{2W}\right)dx\,dy\,dz \\ &\qquad \text{for } m=1,2,3\ldots,n=1,2,3,\ldots\ p=1,2,3\ldots \end{aligned}$$

Since

$$\lim_{t\to\infty} e^{-k\left(\frac{m^2\pi^2}{L^2}+\frac{(2p-1)^2\pi^2}{4W^2}\right)t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-k\left(\frac{m^2\pi^2}{L^2}+\frac{n^2\pi^2}{H^2}+\frac{(2p-1)^2\pi^2}{4W^2}\right)t} = 0,$$

it follows that:

$$\lim_{t\to\infty} u(x,y,z,t) = 0.$$

7.3.4.a. (15pts) Consider the wave equation for a vibrating rectangular membrane $0 < x < L$ and $0 < y < H$:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$

with ICs $u(x,y,0)=0$ and $\frac{\partial u}{\partial t}(x,y,0)=f(x,y)$. The BCs are given by:

$$u(0,y,t)=0, \quad u(L,y,t)=0, \quad \frac{\partial u}{\partial y}(x,0,t)=0, \quad \frac{\partial u}{\partial y}(x,H,t)=0.$$

With separation of variables and assuming c^2 is a constant, we take $u(x,y,t) = \phi(x)g(y)h(t)$, so

$$h''\phi g = c^2(\phi'' gh + g''\phi h), \qquad \text{or} \qquad \frac{h''}{c^2 h} = \frac{\phi''}{\phi} + \frac{g''}{g} = -\lambda.$$

The differential equation in t is $h'' + c^2\lambda h = 0$, which has the solution:

$$h(t) = c_1 \cos\left(c\sqrt{\lambda}t\right) + c_2 \sin\left(c\sqrt{\lambda}t\right),$$

provided $\lambda > 0$.

A second separation of variables gives:

$$\frac{\phi''}{\phi} = -\lambda - \frac{g''}{g} = -\mu.$$

This gives the first SL-problem:

$$\phi'' + \mu\phi = 0, \qquad \text{with} \qquad \phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

This is a standard SL-problem with Dirichlet BC problem, which has eigenvalues and eigenfunctions:

$$\mu_m = \frac{m^2\pi^2}{L^2} \qquad \text{and} \qquad \phi_m(x) = \sin\left(\frac{m\pi x}{L}\right), \quad m = 1, 2, 3...$$

We already showed that $\mu \leq 0$ gives only trivial solutions.

From above we have:

$$-\frac{g''}{g} - \lambda = -\mu_m \qquad \text{or} \qquad g'' + (\lambda - \mu_m)g = g'' + \nu g = 0.$$

The second SL-problem is $g'' + \nu g = 0$ with $g'(0) = 0$ and $g'(H) = 0$. This is a standard Neumann BC problem, which has eigenvalues and eigenfunctions:

$$\begin{aligned} \nu_0 = 0 \qquad & \text{and} \qquad g_0(y) = 1, \\ \nu_n = \frac{n^2\pi^2}{H^2} \qquad & \text{and} \qquad g_n(y) = \cos\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, 3... \end{aligned}$$

We already showed that $\nu < 0$ gives only trivial solutions.

Combining our results, we have:

$$\lambda_{mn} = \mu_m + \nu_n = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{H^2}.$$

The solution to the differential equation in time becomes:

$$h(t) = A_{mn} \cos\left(c\sqrt{\lambda_{mn}}t\right) + B_{mn} \sin\left(c\sqrt{\lambda_{mn}}t\right),$$

which with the initial displacement being zero gives $A_{mn} = 0$.

The Superposition principle gives the solution:

$$\begin{aligned} u(x,y,t) \quad = \quad & \sum_{m=1}^{\infty} B_{m0} \sin\left(\frac{m\pi x}{L}\right) \sin\left(c\sqrt{\frac{m^2\pi^2}{L^2}}t\right) \\ & + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi y}{H}\right) \sin\left(c\sqrt{\frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{H^2}}t\right). \end{aligned}$$

The initial velocity gives:

$$\begin{aligned}\frac{du(x,y,0)}{dt} &= f(x,y) = \sum_{m=1}^{\infty} c\sqrt{\frac{m^2\pi^2}{L^2}} B_{m0} \sin\left(\frac{m\pi x}{L}\right) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\frac{m^2\pi^2}{L^2}+\frac{n^2\pi^2}{H^2}} B_{nm} \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)\end{aligned}$$

From orthogonality, the Fourier coefficients become:

$$\begin{aligned}B_{m0} &= \frac{L\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)dx\,dy}{cm\pi\int_0^H\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx\,dy} \\ &= \frac{2}{cm\pi H}\int_0^H\int_0^L f(x,y)\sin\left(\frac{n\pi x}{L}\right)dx\,dy,\end{aligned}$$

for $m = 1, 2, 3...$ and

$$\begin{aligned}B_{mn} &= \frac{\int_0^H\int_0^L f(x,y)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)dx\,dy}{c\sqrt{\frac{m^2\pi^2}{L^2}+\frac{n^2\pi^2}{H^2}}\int_0^H\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)\cos^2\left(\frac{n\pi y}{H}\right)dx\,dy} \\ &= \frac{4}{LHc\sqrt{\frac{m^2\pi^2}{L^2}+\frac{n^2\pi^2}{H^2}}}\int_0^H\int_0^L f(x,y)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right)dx\,dy,\end{aligned}$$

for $m = 1, 2, 3...$ and $m = 1, 2, 3...$

7.3.5. (8pts) a. Below is a PDE describing a vibrating membrane:

$$\frac{\partial u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) - k\frac{\partial u}{\partial t}, \quad k > 0.$$

The term on the left is the acceleration of the membrane. The first term on the right describes the tension on the membrane, while the second term on the right is viscous damping, such as air resistance on the membrane.

b. Assume c^2 is constant and suppose that $u(x,y,t) = f(x)g(y)h(t)$. We can apply the separation of variables method to obtain:

$$h''fg = c^2(f''gh + g''fh) - kh'fg \qquad \text{or} \qquad \frac{h''+kh'}{c^2h} = \frac{f''}{f}+\frac{g''}{g} = -\lambda.$$

This shows there is the ODE in t given by:

$$h'' + kh' + \lambda c^2 h = 0.$$

A second separation of variables gives:

$$\frac{f''}{f} = -\lambda - \frac{g''}{g} = -\mu,$$

which gives the second ODE in x:

$$f'' + \mu f = 0.$$

Finally, we have

$$\frac{g''}{g} = \mu - \lambda = -\nu,$$

which gives the third ODE in y:

$$g'' + \nu g = 0,$$

where $\lambda = \mu + \nu$.

7.5.1.a. (5pts) The vertical displacement of a non-uniform membrane satisfies

$$\frac{\partial u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

where c depends on x and y. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane. Assume we can separate variables with

$$u(x, y, t) = \phi(x, y)h(t).$$

It follows that:

$$h''\phi(x, y) = c^2\nabla^2\phi(x, y)h \qquad \text{or} \qquad \frac{h''}{h} = \frac{c^2\nabla^2\phi(x, y)}{\phi(x, y)} = -\lambda.$$

This leaves the time ODE:

$$h'' + \lambda h = 0,$$

and the space ODE in (x, y):

$$\nabla^2\phi + \frac{\lambda}{c^2}\phi = 0,$$

where $\phi(x, y) = 0$ on the boundary. This last problem is our eigenvalue problem. From the form of the problem it is easy to see that the weighting function is:

$$\sigma(x, y) = \frac{1}{c^2(x, y)}.$$

7.5.2.a. (7pts) From the previous exercise, we are considering the eigenvalue problem:

$$\nabla^2\phi + \frac{\lambda}{c^2}\phi = 0,$$

with $\phi(x, y) = 0$ on the boundary. We want to prove that distinct eigenfunctions are orthogonal. Consider two eigenfunctions $\phi_m(x, y)$ and $\phi_n(x, y)$ with distinct eigenvalues λ_m and λ_n. Green's theorem gives:

$$\iint_R \left[\phi_m\nabla^2\phi_n - \phi_n\nabla^2\phi_m\right] dx\, dy = \oint_{\partial R} \left[\phi_m\nabla\phi_n - \phi_n\nabla\phi_m\right] \cdot \mathbf{n}\, ds = 0,$$

where the line integral is zero because the eigenfunctions are zero on the boundary of the region. Since

$$\nabla^2\phi_m = -\frac{\lambda_m}{c^2}\phi_m \qquad \text{and} \qquad \nabla^2\phi_n = -\frac{\lambda_n}{c^2}\phi_n,$$

we can write

$$\iint_R \left[\phi_m \nabla^2 \phi_n - \phi_n \nabla^2 \phi_m\right] dx\, dy = -\iint_R \left[\phi_m \phi_n \frac{\lambda_n}{c^2} - \phi_m \phi_n \frac{\lambda_m}{c^2}\right] dx\, dy = 0.$$

It follows that

$$(\lambda_m - \lambda_n) \iint_R \left[\frac{\phi_m \phi_n}{c^2}\right] dx\, dy = 0.$$

Since the eigenvalues are distinct $\lambda_m - \lambda_n \neq 0$, we have the orthogonality:

$$\iint_R \phi_m \phi_n \sigma(x,y)\, dx\, dy = 0,$$

where

$$\sigma(x,y) = \frac{1}{c^2(x,y)}.$$