5.5.1. (10pts) A Sturm-Liouville problem is self-adjoint when

$$\int_a^b [uL(v) - vL(u)]dx = 0,$$

which occurs when

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0.$$

c. If $\phi'(0) - h\phi(0) = 0$ and $\phi'(L) = 0$, then we have $u'(0) = hu(0)$, $u'(L) = 0$, $v'(0) = hv(0)$, and $v'(L) = 0$. Substituting the B.C.s into the conditions above, we have

$$p(L)\left[u(L)v'(L) - v(L)u'(L)\right] - p(0)\left[u(0)v'(0) - v(0)u'(0)\right] = p(L)\left[u(L)\cdot 0 - v(L)\cdot 0\right] - p(0)\left[u(0)hv(0) - v(0)hu(0)\right] = 0.$$

Thus, these B.C.s give the operator L being self-adjoint.

d. If $\phi(a) = \phi(b)$ and $p(a)\phi'(a) = p(b)\phi'(b)$, then we have $u(a) = u(b)$, $p(a)u'(a) = p(b)u'(b)$, $v(a) = v(b)$, and $p(a)v'(a) = p(b)v'(b)$. Substituting the B.C.s into the conditions above, we have

$$\begin{aligned} p(b)\left[u(b)v'(b) - v(b)u'(b)\right] - p(a)\left[u(a)v'(a) - v(a)u'(a)\right] &= u(b)[p(b)v'(b)] - v(b)[p(b)u'(b)] - u(a)[p(a)v'(a)] + v(a)[p(a)u'(a)], \\ &= u(a)[p(a)v'(a)] - v(a)[p(a)u'(a)] - u(a)[p(a)v'(a)] + v(a)[p(a)u'(a)], \\ &= 0. \end{aligned}$$

Thus, these B.C.s give the operator L being self-adjoint.

5.5.5. (10pts) Consider the operator $L = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9$.

a. Apply the operator to e^{rx}, then we have

$$\begin{aligned} L(e^{rx}) &= \frac{d^2}{dx^2}(e^{rx}) + 6\frac{d}{dx}(e^{rx}) + 9(e^{rx}), \\ &= r^2e^{rx} + 6re^{rx} + 9e^{rx} = (r^2 + 6r + 9)e^{rx} = (r+3)^2e^{rx}. \end{aligned}$$

b. If $L(y) = 0$ is a second order DE, then for $y = e^{rx}$ we have $L(y) = (r+3)^2y = 0$ (Part a). For nontrivial solutions, $r = -3$, and $y = e^{-3x}$ is a solution.

c. Consider $z(x, r)$, then $L(z) = \frac{d^2z}{dx^2} + 6\frac{dz}{dx} + 9z$, so

$$\begin{aligned} \frac{\partial}{\partial r}[L(z)] &= \frac{\partial}{\partial r}\left(\frac{d^2z}{dx^2}\right) + 6\frac{\partial}{\partial r}\left(\frac{dz}{dx}\right) + 9\frac{\partial z}{\partial r}, \\ &= z_{xxr} + 6z_{xr} + 9z_r, \\ L(z_r) &= \frac{d^2z_r}{dx^2} + 6\frac{dz_r}{dx} + 9z_r, \\ &= z_{rxx} + 6z_{rx} + 9z_r. \end{aligned}$$

Assuming that all the partial derivatives are continuous, we have $z_{rxx} = z_{xxr}$ and $z_{rx} = z_{xr}$, so

$$\frac{\partial}{\partial r}L(z) = L\left(\frac{\partial z}{\partial r}\right).$$

d. Let $z = e^{rx}$, then $\frac{\partial z}{\partial r} = xe^{rx}$. From Part c, we have

$$L\left(xe^{rx}\right) = \frac{\partial}{\partial r}\left[L\left(e^{rx}\right)\right] = \frac{\partial}{\partial r}\left[(r+3)^2 e^{rx}\right].$$

It follows that

$$L\left(xe^{rx}\right) = 2(r+3)e^{rx} + x(r+3)^2 e^{rx} = e^{rx}(r+3)\left[2 + x(r+3)\right].$$

e. From Part d, we have $L\left(xe^{rx}\right) = e^{rx}(r+3)\left[2+x(r+3)\right]$. From this expression it is clear that for all x, if $r = -3$, we have

$$L\left(xe^{-3x}\right) = 0,$$

so $y(x) = xe^{-3x}$ is another solution to our linear operator L.

5.5.8. (15pts) Consider the 4^{th} order linear operator (often in beam problems)

$$L = \frac{d^4}{dx^4}.$$

a. We expand this operator

$$\begin{aligned} uL(v) - vL(u) &= u \cdot v^{(4)} - v \cdot u^{(4)}, \\ &= uv^{(4)} + u'v^{(3)} - u'v^{(3)} - u''v'' + u''v'' + u^{(3)}v' - u^{(3)}v' - u^{(4)}v, \\ &= \left(uv^{(3)}\right)' - \left(u'v''\right)' + \left(u''v'\right)' - \left(u^{(3)}v\right)' = \left[uv^{(3)} - u'v'' + u''v' - u^{(3)}v\right]', \\ &= \frac{d}{dx}\left[uv^{(3)} - u'v'' + u''v' - u^{(3)}v\right], \end{aligned}$$

which is an exact differential.

b. We use the Fundamental Theorem of Calculus to integrate and evaluate this exact differential:

$$\begin{aligned} \int_0^1 \left[uL(v) - vL(u)\right] dx &= \int_0^1 \left[\frac{d}{dx}(uv^{(3)} - u'v'' + u''v' - u^{(3)}v)\right] dx, \\ &= \left(uv^{(3)} - u'v'' + u''v' - u^{(3)}v\right)\Big|_0^1. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_0^1 \left[uL(v) - vL(u)\right] dx &= u(1)v^{(3)}(1) - u'(1)v''(1) + u''(1)v'(1) - u^{(3)}(1)v(1) \\ &\quad - u(0)v^{(3)}(0) + u'(0)v''(0) - u''(0)v'(0) + u^{(3)}(0)v(0). \end{aligned}$$

c. If u and v are any two functions satisfying the B.C.'s, we have

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 0, \quad u''(1) = 0,$$

$$v(0) = 0, \quad v'(0) = 0, \quad v(1) = 0, \quad v''(1) = 0.$$

the expression in Part b becomes:

$$\begin{aligned}\int_0^1 [uL(v) - vL(u)]\,dx &= 0\cdot v^{(3)}(1) - u'(1)\cdot 0 + 0\cdot v'(1) - u^{(3)}(1)\cdot 0\\ &\quad -0\cdot v^{(3)}(0) + 0\cdot v''(0) - u''(0)\cdot 0 + u^{(3)}(0)\cdot 0 \;=\; 0.\end{aligned}$$

Thus, we have that L is self-adjoint with

$$\int_0^1 [uL(v) - vL(u)]\,dx = 0.$$

d. Very clearly there are many other B.C.'s that result in this operator being self-adjoint. The most common are "pinned" B.C.'s, where $\phi(0) = 0$ or $\phi(1) = 0$, or "clamped" B.C.'s, where $\phi'(0) = 0$ or $\phi'(1) = 0$, or "free pivot (no force)" B.C.'s, where $\phi''(0) = 0$ or $\phi''(1) = 0$. Obviously, four appropriate conditions must be satisfied for L to be self-adjoint.

e. Let λ_n be eigenvalues with corresponding eigenfunctions ϕ_n and assume the B.C.'s of Part c for the eigenvalue problem:

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0.$$

Let $\lambda_n \neq \lambda_m$ have associated eigenfunctions ϕ_n and ϕ_m. From the B.C.'s, we have:

$$\begin{aligned}\int_0^1 [\phi_n\cdot L(\phi_m) - \phi_m\cdot L(\phi_n)]\,dx &= 0,\\ \text{or}\quad \int_0^1 \left[\phi_n\cdot\phi_m^{(4)} - \phi_m\cdot\phi_n^{(4)}\right]dx &= 0.\end{aligned}$$

However, since $\frac{d^4\phi}{dx^4} = -\lambda e^x\phi$, it follows that

$$\begin{aligned}\int_0^1 [\phi_n(-\lambda_m e^x\phi_m) - \phi_m(-\lambda_n e^x\phi_n)]\,dx &= 0,\\ (\lambda_n - \lambda_m)\int_0^1 \phi_m\phi_n e^x\,dx &= 0.\end{aligned}$$

Since λ_m and λ_n are distinct eigenvalues, $\int_0^1 \phi_m\phi_n e^x\,dx = 0$, which shows that the eigenfunctions, ϕ_i are orthogonal with respect to the weighting function $\sigma(x) = e^x$.

5.5.11. (15pts) Consider the linear operator $L = p(x)\frac{d^2}{dx^2} + r(x)\frac{d}{dx} + q(x)$, we examine:

$$\int_a^b v\cdot L(u)\,dx = \int_a^b (vpu'' + vru' + vqu)\,dx = \int_a^b u''vp\,dx + \int_a^b u'vr\,dx + \int_a^b uvq\,dx.$$

Using integration by parts on the first integral gives:

$$\begin{aligned}\int_a^b u''vp\,dx &= u'vp\big|_a^b - \int_a^b u'(vp' + v'p)\,dx,\\ &= [u'vp - u(vp' + v'p)]\big|_a^b + \int_a^b u(vp'' + 2v'p' + v''p)dx.\end{aligned}$$

Using integration by parts on the second integral gives:

$$\int_a^b u'vr\ dx = uvr|_a^b - \int_a^b u(vr' + v'r)\ dx.$$

We combine these results to give:

$$\begin{aligned}
\int_a^b v \cdot L(u)\ dx &= \int_a^b u\left[p\frac{d^2v}{dx^2} + \left(2\frac{dp}{dx} - r\right)\frac{dv}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)v\right]dx \\
&\quad - \left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) + uv\left(\frac{dp}{dx} - r\right)\right|_a^b, \\
&= \int_a^b uL^*(v)\,dx - \left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) + uv\left(\frac{dp}{dx} - r\right)\right|_a^b, \\
&= \int_a^b uL^*(v)dx - H(x)|_a^b,
\end{aligned}$$

where

$$L^* = p\frac{d^2}{dx^2} + \left(2\frac{dp}{dx} - r(x)\right)\frac{d}{dx} + \left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q(x)\right)$$

and

$$H(x) = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) + uv\left(\frac{dp}{dx} - r\right)$$

Thus, we can write:

$$\int_a^b [uL^*(v) - vL(u)]\,dx = H(x)|_a^b.$$

From these expressions we find that the operator L is self-adjoint ($L = L^*$) if and only if

$$2\frac{dp}{dx} - r(x) = r(x) \qquad \text{or} \qquad \frac{dp}{dx} = r(x)$$

and $H(b) - H(a) = 0$. Since $p' = r$, the latter condition reduces to

$$p(b)\left(u(b)v'(b) - v(b)u'(b)\right) - p(a)\left(u(a)v'(a) - v(a)u'(a)\right) = 0.$$

b. Assume that the B.C.'s on u satisfy:

$$u(0) = 0 \qquad \text{and} \qquad \frac{du}{dx}(L) + u(L) = 0 \quad \text{or} \quad \frac{du}{dx}(L) = -u(L),$$

then for self-adjointness we need $H(L) - H(0) = 0$ (assuming $p' = r$). These conditions imply:

$$\begin{aligned}
H(L) - H(0) &= p(L)\left(u(L)v'(L) - v(L)u'(L)\right) - p(0)\left(u(0)v'(0) - v(0)u'(0)\right) \\
&= p(L)u(L)\left(v'(L) + v(L)\right) - p(0)\left(v(0)u'(0)\right) = 0.
\end{aligned}$$

This condition will hold if

$$v(0) = 0 \qquad \text{and} \qquad v'(L) + v(L) = 0.$$

5.8.5. a. (8pts) Consider the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with B.C.'s and I.C's

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = -hu(L,t), \quad \text{and} \quad u(x,0) = f(x).$$

Start with separation of variables, $u(x,t) = \phi(x)g(t)$, so

$$\phi g' = k\phi'' g \qquad \text{or} \qquad \frac{g'}{kg} = \frac{\phi''}{\phi} = -\lambda.$$

Let $h > 0$ and consider the SL problem:

$$\phi'' + \lambda\phi = 0, \qquad \text{with B.C.'s} \quad \phi'(0) = 0 \quad \text{and} \quad \phi'(L) + h\phi(L) = 0.$$

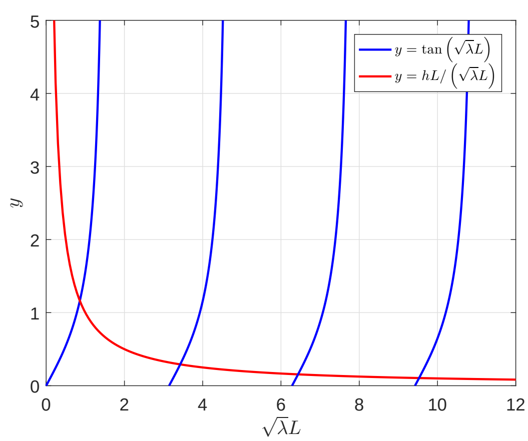
If $\lambda = -\alpha^2 < 0$, then $\phi(x) = c_1\cosh(\alpha x) + c_2\sinh(\alpha x)$. If $\phi'(0) = 0$, then $c_2 = 0$. For $\phi'(L) + h\phi(L) = 0$, then $c_1(\alpha\sinh(\alpha L) + h\cosh(\alpha L)) = 0$, which implies $c_1 = 0$ (for $h > 0$) and only the trivial solution exists. Similarly, if $\lambda = 0$, then $\phi(x) = c_1 x + c_2$. With $\phi'(0) = 0$, then $c_1 = 0$. The B.C. $\phi'(L) + h\phi(L) = hc_2 = 0$ shows that $c_2 = 0$, which again leaves only the trivial solution.

If $\lambda = \alpha^2 > 0$, then $\phi(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$. If $\phi'(0) = 0$, then $c_2 = 0$. For $\phi'(L) + h\phi(L) = 0$, then $c_1(-\alpha\sin(\alpha L) + h\cos(\alpha L)) = 0$. This has nontrivial solutions when $\tan(\alpha L) = \frac{hL}{\alpha L}$. Thus, we have eigenfunctions:

$$\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right),$$

where the eigenvalues λ_n solve the transcendental equation:

$$\tan\left(\sqrt{\lambda}L\right) = \frac{hL}{\sqrt{\lambda}L}.$$



The temporal equation is $g' = -k\lambda_n g$, which has the solution, $g_n(t) = a_n e^{-k\lambda_n t}$.

The superposition principle gives:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t} \cos\left(\sqrt{\lambda_n} x\right).$$

Applying the I.C. yields:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \cos \sqrt{\lambda_n} x = f(x).$$

By the orthogonality of $\phi(x)$, we obtain the Fourier coefficients:

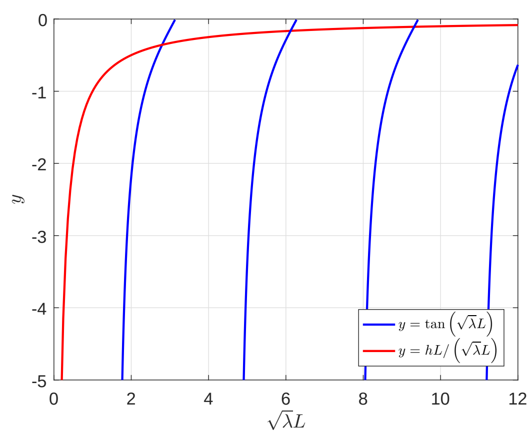
$$a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} = \frac{\int_0^L f(x)\cos\left(\sqrt{\lambda_n}x\right)dx}{\int_0^L \cos^2\left(\sqrt{\lambda_n}x\right)dx}.$$

b. (12pts) If $h < 0$ (non-physical case), then similar to Part a, there are eigenfunctions:

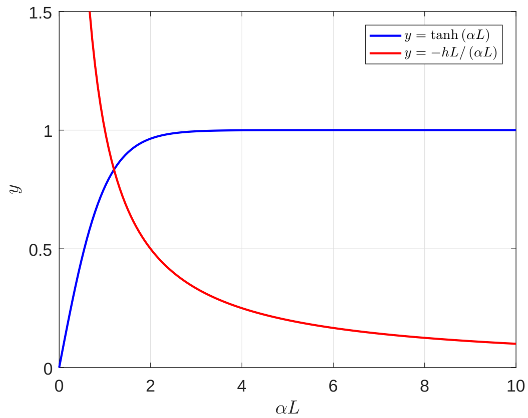
$$\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right),$$

where the eigenvalues λ_n solve the transcendental equation:

$$\tan\left(\sqrt{\lambda}L\right) = \frac{hL}{\sqrt{\lambda}L}.$$



If $\lambda = -\alpha^2 < 0$, then $\phi(x) = c_1\cosh(\alpha x) + c_2\sinh(\alpha x)$. If $\phi'(0) = 0$, then $c_2 = 0$. For $\phi'(L) + h\phi(L) = 0$, then $c_1(\alpha\sinh(\alpha L) + h\cosh(\alpha L)) = 0$. If $\alpha\sinh(\alpha L) + h\cosh(\alpha L) = 0$, then $\tanh(\alpha L) = -\frac{hL}{\alpha L}$. This equation has a unique solution, producing a negative eigenvalue. The graph below shows a typical intersection from the equation above.



Thus, we have one negative eigenvalue λ_{-1} with corresponding eigenfunction:

$$\phi_{-1}(x) = \cosh\left(\sqrt{-\lambda_{-1}}x\right).$$

As before, the temporal problem has the solution:

$$g(t) = e^{-\lambda_n kt}, \qquad n = -1, 1, 2, ...$$

Numerically, we find the first five eigenvalues, λ_{-1}, λ_1, λ_2, λ_3, and λ_4. These eigenvalues are

$$\begin{array}{ccc} \lambda_{-1} = -1.439229 & \lambda_1 = 7.830964 & \lambda_2 = 37.469707 \\ \lambda_3 = 86.822635 & \lambda_4 = 155.911544 & \end{array}$$

We apply the superposition principle to obtain the solution:

$$u(x,t) = a_{-1}e^{-\lambda_{-1}kt}\cosh\left(\sqrt{-\lambda_{-1}}x\right) + \sum_{n=1}^{\infty} a_n e^{-\lambda_n kt}\cos\left(\sqrt{\lambda_n}x\right).$$

The I.C. gives:

$$u(x,0) = f(x) = a_{-1}\cosh\left(\sqrt{-\lambda_{-1}}x\right) + \sum_{n=1}^{\infty} a_n \cos\left(\sqrt{\lambda_n}x\right).$$

From the orthogonality of the eigenfunctions, the Fourier coefficients satisfy:

$$a_n = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} = \begin{cases} \dfrac{\int_0^L f(x)\cosh\sqrt{-\lambda_{-1}}x\ dx}{\int_0^L \cosh^2\sqrt{-\lambda_{-1}}x\,dx}, & n = -1, \\ \dfrac{\int_0^L f(x)\cos\sqrt{\lambda_n}x\ dx}{\int_0^L \cos^2\sqrt{\lambda_n}x\,dx}, & n \geq 1. \end{cases}$$

5.8.8. a. (5pts) Consider the BVP:

$$\phi'' + \lambda\phi = 0, \qquad \text{with} \quad \phi(0) - \phi'(0) = 0 \quad \text{and} \quad \phi(1) + \phi'(1) = 0.$$

The Rayleigh quotient gives

$$\lambda = \frac{-\phi\frac{d\phi}{dx}\Big|_0^1 + \int_0^1\left(\frac{d\phi}{dx}\right)^2 dx}{\int_0^1 \phi^2 dx}.$$

However, since $\phi'(1) = -\phi(1)$ and $\phi'(0) = \phi(0)$, we see that

$$-\phi\phi'\Big|_0^1 = -\phi(1)\phi'(1) + \phi(0)\phi'(0) = \phi^2(1) + \phi^2(0),$$

so it follows that

$$\lambda = \frac{\phi^2(1) + \phi^2(0) + \int_0^1 (\phi')^2 dx}{\int_0^1 \phi^2 dx} \geq 0.$$

If $\lambda = 0$, then the expression above implies that $\phi' = 0$ or ϕ is constant. The B.C.'s show that if ϕ is constant, then $\phi(x) \equiv 0$, so is not an eigenfunction. Thus, it follows that $\lambda > 0$.

b. (5pts) Let $L = \frac{d^2}{dx^2}$ and ϕ_n and ϕ_m eigenfunctions with eigenvalues λ_n and λ_m for $n \neq m$. It follows that

$$L[\phi_n] + \lambda_n\phi_n = 0 \qquad \text{and} \qquad L[\phi_m] + \lambda_m\phi_m = 0,$$

so

$$\int_0^1 (\phi_m(L[\phi_n] + \lambda_n\phi_n) - \phi_n(L[\phi_m] + \lambda_m\phi_m))\, dx = 0,$$

or

$$\int_0^1 (\phi_m L[\phi_n] - \phi_n L[\phi_m] + (\lambda_n - \lambda_m)\phi_n\phi_m)\, dx = 0.$$

So integrating by parts gives

$$\left[\phi_m \frac{d}{dx}\phi_n - \phi_n \frac{d}{dx}\phi_m\right|_0^1 + (\lambda_n - \lambda_m)\int_0^1 \phi_n\phi_m\ dx = 0,$$

or

$$\phi_m(1)\phi'_n(1) - \phi_n(1)\phi'_m(1) - \phi_m(0)\phi'_n(0) + \phi_n(0)\phi'_m(0) + (\lambda_n - \lambda_m)\int_0^1 \phi_n\phi_m\ dx = 0.$$

Since the B.C.'s satisfy $\phi'(1) = -\phi(1)$ and $\phi'(0) = \phi(0)$, the expression above reduces to

$$(\lambda_n - \lambda_m)\int_0^1 \phi_n\phi_m\ dx = 0 \qquad \text{or} \qquad \int_0^1 \phi_n\phi_m\ dx = 0.$$

Therefore, ϕ_n and ϕ_m are orthogonal.

c. (9pts) We solve $\phi'' + \lambda\phi = 0$ (with $\lambda > 0$). The general solution is

$$\phi(x) = c_1\cos\left(\sqrt{\lambda}x\right) + c_2\sin\left(\sqrt{\lambda}x\right), \quad \text{so} \quad \phi'(x) = -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right).$$

The B.C. $\phi(0) - \phi'(0) = 0$ gives $c_1 - c_2\sqrt{\lambda} = 0$ or $c_1 = c_2\sqrt{\lambda}$. The other B.C. $\phi(1) + \phi'(1) = 0$ gives:

$$c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) + c_2\sin\left(\sqrt{\lambda}\right) - c_2\lambda\sin\left(\sqrt{\lambda}\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) = 0.$$

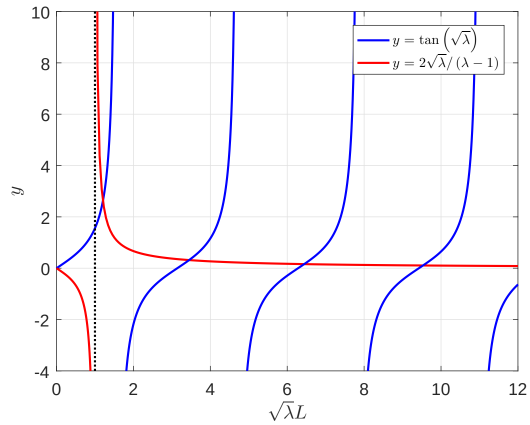
Combining terms gives

$$c_2\left[2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) + (1-\lambda)\sin\left(\sqrt{\lambda}\right)\right] = 0,$$

which for nontrivial solutions yields $2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) + (1-\lambda)\sin\left(\sqrt{\lambda}\right) = 0$ or

$$\tan\left(\sqrt{\lambda}\right) = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Below is a graph of the right and left hand functions of $\sqrt{\lambda}$ with intersections producing the square root of eigenvalues.



If $f(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{(\lambda-1)}$ and $g(\sqrt{\lambda}) = \tan\sqrt{\lambda}$ are the right and left hand functions for the eigenvalue equation, then $f(x) = g(x)$ at $x = 0$, which is not an eigenvalue. All subsequent intersections occur after the vertical asymptote at $x = 1$. We have $0 < \sqrt{\lambda_1} < \frac{\pi}{2}$, $\pi < \sqrt{\lambda_2} < \frac{3\pi}{2}$, $2\pi < \sqrt{\lambda_3} < \frac{5\pi}{2}$,... Furthermore, we readily see that $\lim_{\sqrt{\lambda}\to\infty} f(\sqrt{\lambda}) = 0$. It follows that

$$(n-1)\pi < \sqrt{\lambda_n} < \frac{(2n-1)\pi}{2}, \quad n \geq 1,$$

and for large n

$$\sqrt{\lambda_n} \simeq (n-1)\pi.$$

d. (6pts) Consider the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad \text{with I.C.} \quad u(x,0) = f(x),$$

and B.C.'s $u(0,t) - u_x(0,t) = 0$ and $u(1,t) + u_x(1,t) = 0$. Separation of variables with $u(x,t) = \phi(x)h(t)$ gives:

$$\frac{h'}{kh} = \frac{\phi''}{\phi} = -\lambda.$$

This produces the SL problem:

$$\phi'' + \lambda\phi = 0, \qquad \text{with} \quad \phi(0) - \phi'(0) = 0 \quad \text{and} \quad \phi(1) + \phi'(1) = 0,$$

where λ satisfies the equation $\tan\left(\sqrt{\lambda}\right) = \frac{2\sqrt{\lambda}}{\lambda-1}$. From Part c, we produced the eigenfunctions:

$$\phi_n(x) = \sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right) + \sin\left(\sqrt{\lambda_n}x\right),$$

which were orthogonal according to Part b.

The time-dependent problem is readily solved:

$$h' + \lambda_n kh = 0, \qquad \text{so} \qquad h(t) = ce^{-\lambda_n kt}.$$

The superposition principle gives:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t}\phi_n(x).$$

To satisfy the I.C. we need:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x) = f(x).$$

We multiply by $\phi_m(x)$ and integrate from 0 to 1. Using orthogonality, we obtain:

$$a_m \int_0^1 \phi_m^2(x)\,dx = \int_0^1 f(x)\phi_m(x)\,dx \qquad \text{or} \qquad a_m = \frac{\int_0^1 f(x)\phi_m(x)\,dx}{\int_0^1 \phi_m^2(x)\,dx}.$$

5.8.11. (5pts) Consider the SL problem:

$$\phi'' + 5\phi = -\lambda\phi, \qquad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(\pi) = 0.$$

Let $\mu = \lambda + 5$, then we are solving the SL problem:

$$\phi'' + \mu\phi = 0 \qquad \text{with} \quad \phi(0) = 0 \quad \text{and} \quad \phi(\pi) = 0.$$

We have seen before that this problem has eigenvalues, μ_n, and eigenfunctions $\phi_n(x)$ given by:

$$\mu_n = n^2 \qquad \text{with} \qquad \phi_n(x) = \sin(nx), \quad n = 1, 2, 3, ...$$

However, $\lambda_n = \mu_n - 5 = n^2 - 5$, so the first eigenvalues are

$$\begin{aligned} \lambda_1 &= 1 - 5 = -4, \\ \lambda_2 &= 4 - 5 = -1, \\ \lambda_3 &= 9 - 5 = 4, \\ \lambda_n &> 0, \quad \text{for} \quad n \geq 3. \end{aligned}$$

The negative eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -1$.

WeBWorK 2. a. (5pts) Consider the SL problem ($h > 0$):

$$\phi'' + \lambda\phi = 0, \qquad \text{with} \quad \phi'(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) + h\phi(L) = 0,$$

where $p(x) = 1$, $q(x) = 0$, and $\sigma(x) = 1$. The Rayleigh quotient satisfies:

$$\lambda = \frac{-p\phi\phi'|_0^L + \int_0^L \left(p\,(\phi')^2 - q\phi^2\right) dx}{\int_0^L \phi^2\sigma\,dx}.$$

We use the information on p, q, and σ with $\phi'(L) = -h\phi(L)$ ($h > 0$) to reduce the expression above to

$$\lambda = \frac{-\phi(L)\phi'(L) + \int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx} = \frac{h\phi^2(L) + \int_0^L (\phi')^2\,dx}{\int_0^L \phi^2\,dx} \geq 0.$$

If $\lambda = 0$, then $\phi' = 0$, which implies $\phi(x) = C$. However, $\phi'(L) = 0 = -h\phi(L)$ gives $\phi(x) \equiv 0$, which is not an eigenfunction. Thus, $\lambda > 0$.

c. (5pts) From Problem 5.8.5 above we find the eigenfunctions are:

$$\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right),$$

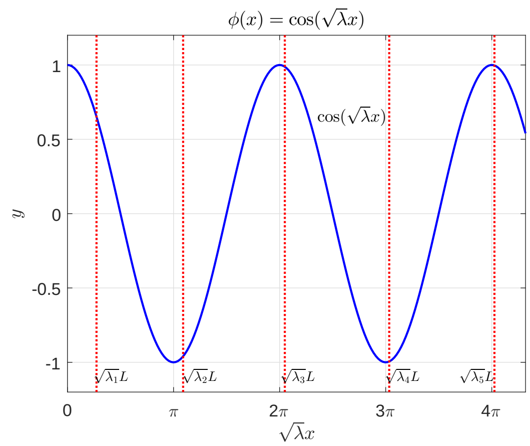
where the eigenvalues λ_n solve the transcendental equation:

$$\tan\left(\sqrt{\lambda}L\right) = \frac{hL}{\sqrt{\lambda}L}.$$

The graph above shows that each eigenvalue lies in an interval:

$$\sqrt{\lambda_n} \in \left(\frac{(n-1)\pi}{L}, \frac{(2n-1)\pi}{2L}\right) \qquad \text{with} \qquad \sqrt{\lambda_n} \to \frac{n\pi}{L}, \quad \text{as} \quad n \to \infty.$$

Below we graph the eigenfunction:



We see that the eigenfunction, $\phi_1(x)$ has no zeros for $x \in [0, \sqrt{\lambda_1}L]$. For the eigenfunction, $\phi_2(x)$ there is one zero for $x \in [0, \sqrt{\lambda_2}L]$. Similarly, we see that the eigenfunction, $\phi_3(x)$ there are two zeros for $x \in [0, \sqrt{\lambda_3}L]$. Asymptotically, we have $\sqrt{\lambda_n} \to \frac{n\pi}{L}$, and we know that $\phi_n(x) \approx \cos\left(\frac{n\pi x}{L}\right)$ has $n-1$ zeros for $x \in [0, L]$, which was the desired result.