HW 5

4.4.1. Consider vibrating strings of uniform density ρ_0 and tension T_0.

-15 4 *(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?

6 *(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.

5 (c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

4.4.9 From (4.4.1), derive conservation of energy for a vibrating string,

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$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L, \tag{4.4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

4.4.10. What happens to the total energy E of a vibrating string (see Exercise 4.4.9)

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4 (a) If $u(0,T) = 0$ and $u(L,t) = 0$

4 (b) If $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0$

6 (c) If $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = -\gamma u(L,t)$ with $\gamma > 0$

3 (d) If $\gamma < 0$ in part (c)

5.3.2. Consider

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u + \beta \frac{\partial u}{\partial t}.$$

15 3 (a) Give a brief physical interpretation. What signs must α and β have to be physical?

6 (b) Allow ρ, α, β to be functions of x. Show that separation of variables works only if $\beta = c\rho$, where c is a constant.

6 (c) If $\beta = c\rho$, show that the spatial equation is a Sturm-Liouville differential equation. Solve the time equation.

*5.3.3. Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

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Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

HW 5 (cont).

5.3.9. Consider the eigenvalue problem

$$x^2 \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = 0, \quad \text{and} \quad \phi(b) = 0. \quad (5.3.10)$$

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3 (a) Show that multiplying by $1/x$ puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

5 (b) Show that $\lambda \geq 0$.

5 *(c) Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.

4 (d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

3 (e) Show that the nth eigenfunction has $n-1$ zeros.

5.4.5. Consider

$$\rho\frac{\partial u^2}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u,$$

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where $\rho(x) > 0, \alpha(x) < 0$, and T_0 is constant, subject to

$$\begin{array}{ll} u(0,t) = 0 & u(x,0) = f(x) \\ u(L,t) = 0 & \frac{\partial u}{\partial t}(x,0) = g(x). \end{array}$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem.