8.2.2.b. (7 pts) Consider the nonhomogeneous heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

with the following IC and nonhomogeneous BCs

$$u(x,0) = f(x) \qquad \text{with} \quad u(0,t) = A(t) \quad \text{and} \quad u_x(L,t) = B(t).$$

Let $u(x,t) = v(x,t) + r(x,t)$, then $v_t + r_t = kv_{xx} + kr_{xx} + Q$ with IC $v(x,0) + r(x,0) = f(x)$ and BCs $v(0,t) + r(0,t) = A(t)$ and $v_x(L,t) + r_x(L,t) = B(t)$, where $r(x,t)$ is a reference function. We want $r(x,t)$ to be a simple function that satisfies the nonhomogeneous BCs, so $r(0,t) = A(t)$ and $r_x(L,t) = B(t)$. Take

$$r(x,t) = A(t) + xB(t),$$

then this satisfies the nonhomogeneous BCs.

The $v(x,t)$ problem becomes:

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - A'(t) - xB'(t),$$

since $r_{xx} = 0$. The IC and BCs become:

$$v(x,0) = f(x) - A(0) - xB(0) \qquad \text{with} \quad v(0,t) = 0 \quad \text{and} \quad v_x(L,t) = 0.$$

8.2.5. (15 pts) Consider 2D heat equation:

$$\frac{\partial u}{\partial t} = k\nabla^2 u = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right),$$

with BC $u(a,\theta,t) = g(\theta)$ and IC $u(r,\theta,0) = f(r,\theta)$. There is the implicit (homogeneous) BC is boundedness at the origin or $|u(0,\theta,t)| < \infty$ and periodicity in θ.

The equilibrium solution $u_E(r,\theta)$ satisfies $\nabla^2 u_E = 0$ with $u_E(a,\theta) = g(\theta)$ and $|u(0,\theta)| < \infty$. If we let $u_E(r,\theta) = \phi(r)g(\theta)$, then separation gives:

$$\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\frac{g''}{g} = \nu.$$

This gives the SL problem $g'' + \nu g = 0$ with periodic BCs ($g(-\pi) = g(\pi)$ and $g'(-\pi) = g'(\pi)$), which we have seen before has e.v.s $\nu_n = n^2$, $n = 0, 1, 2, ...$ and e.f's $g_0(\theta) = 1$ and $g_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$. The other ODE is $r(r\phi')' - n^2\phi = 0$, which has solutions:

$$\phi_0(r) = b_0 + c_0\ln(r) \qquad \text{and} \qquad \phi_n(r) = b_n r^n + c_n r^{-n}.$$

The boundedness at $r = 0$ implies $c_i = 0$. Superposition gives:

$$u_E(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)r^n.$$

The IC gives:

$$u_E(a,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) a^n = g(\theta),$$

where the Fourier coefficients satisfy:

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\theta, \\ A_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\, d\theta, \\ B_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\, d\theta. \end{aligned}$$

Let $u(r,\theta,t) = v(r,\theta,t) + u_E(r,\theta)$, then $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}$ and $\nabla^2 u = \nabla^2 v + \nabla^2 u_E = \nabla^2 v$. It follows that v satisfies the heat equation:

$$\frac{\partial v}{\partial t} = k\nabla^2 v = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right),$$

with BC $v(a,\theta,t) = 0$ and IC $v(r,\theta,0) = f(r,\theta) - u_E(r,\theta)$. This is a problem solved before with:

$$\begin{aligned} v(r,\theta,t) &= \sum_{n=1}^{\infty} A_{0n} J_0\left(\sqrt{\lambda_{0n}}r\right) e^{-k\lambda_{0n}t} \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right) J_m\left(\sqrt{\lambda_{mn}}r\right) e^{-k\lambda_{mn}t}, \end{aligned}$$

where $\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$ and z_{mn} is the n^{th} zero of $J_m(z)$.

The IC gives:

$$\begin{aligned} v(r,\theta,0) &= \sum_{n=1}^{\infty} A_{0n} J_0\left(\sqrt{\lambda_{0n}}r\right) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right) J_m\left(\sqrt{\lambda_{mn}}r\right) \\ &= f(r,\theta) - u_E(r,\theta) \;=\; F(r,\theta). \end{aligned}$$

The double Fourier series coefficients satisfy:

$$\begin{aligned} A_{0n} &= \frac{\int_{-\pi}^{\pi}\int_0^a F(r,\theta) J_0\left(\sqrt{\lambda_{0n}}r\right) r\, dr\, d\theta}{2\pi \int_0^a J_0^2\left(\sqrt{\lambda_{0n}}r\right) r\, dr} \\ A_{mn} &= \frac{\int_{-\pi}^{\pi}\int_0^a F(r,\theta)\cos(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r\, dr\, d\theta}{\pi \int_0^a J_m^2\left(\sqrt{\lambda_m}r\right) r\, dr} \\ B_{mn} &= \frac{\int_{-\pi}^{\pi}\int_0^a F(r,\theta)\sin(m\theta) J_m\left(\sqrt{\lambda_{mn}}r\right) r\, dr\, d\theta}{\pi \int_0^a J_m^2\left(\sqrt{\lambda_m}r\right) r\, dr} \end{aligned}$$

The solution is $u(r,\theta,t) = v(r,\theta,t) + u_E(r,\theta)$.

8.3.1.a. (10 pts) Consider the nonhomogeneous heat problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

with the IC $u(x,0) = f(x)$ and BCs $u(0,t) = 0$ and $u_x(L,t) = 0$. The related homogeneous problem is:

$$\frac{\partial w}{\partial t} = k\frac{\partial^2 w}{\partial x^2} \qquad \text{with} \quad w(0,t) = 0 \quad \text{and} \quad w_x(L,t) = 0.$$

The associated eigenvalues and eigenfunctions for this problem are:

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \qquad \text{and} \qquad \phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad n = 1, 2, ...$$

Let

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

It follows that:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} {a_n}'(t)\phi_n(x) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x).$$

We assume that $Q(x,t)$ has an eigenfunction expansion:

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x), \qquad \text{where} \qquad q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\phi_n(x)\,dx.$$

From the original heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t), \\ \sum_{n=1}^{\infty} {a_n}'(t)\phi_n(x) &= -\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \end{aligned}$$

By orthogonality of $\phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$, we obtain an ODE for the Fourier coefficients $a_n(t)$:

$${a_n}'(t) + \lambda_n a_n(t) = q_n(t).$$

The ICs for this problem give the ICs for the ODE in $a_n(t)$, since

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x),$$

where

$$a_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)\,dx.$$

The ODE in $a_n(t)$ is a linear nonhomogeneous equation with an integrating factor of $e^{\lambda_n kt}$, and its solution is given by:

$$a_n(t) = a_n(0)e^{-\lambda_n kt} + e^{-\lambda_n kt}\int_0^t q_n(s)e^{\lambda_n ks}\,ds.$$

With these time-dependent Fourier coefficients, the solution satisfies:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

8.4.2. (15 pts) Consider the heat problem with nonhomogeneous BCs:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with IC, $u(x,0) = f(x)$ and BCs $u(0,t) = A$ and $u(L,t) = B$. The eigenfunctions of the related homogeneous problem have the form $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ with eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, ...$

We assume a solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi x}{L}\right).$$

The partial w.r.t. t gives:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} {b_n}'(t)\phi_n(x) = k\frac{\partial^2 u}{\partial x^2}.$$

From the orthogonality of the eigenfunctions (and $\int_0^L \phi_n^2(x) = \frac{L}{2}$), we have:

$$b_n'(t) = \frac{2k}{L}\int_0^L u_{xx}\phi_n(x)\,dx.$$

Apply Green's formula:

$$\int_0^L u\frac{d^2\phi_n}{dx^2} - \phi_n\frac{\partial^2 u}{\partial x^2}\,dx = \left[u\frac{d\phi}{dx} - \phi\frac{du}{dx}\right]\Bigg|_0^L,$$

where $\phi_n(0) = 0$, $\phi_n(L) = 0$, and $\frac{d^2\phi_n}{dx^2} = -\lambda_n\phi_n$. This gives:

$$\begin{aligned}
\int_0^L u_{xx}\phi_n(x)\,dx &= -\lambda_n\int_0^L u\phi_n(x)\,dx - u(L,t)\frac{d\phi}{dx}(L) + u(0,t)\frac{d\phi}{dx}(0), \\
&= -\lambda_n\int_0^L b_n(t)\phi_n^2(x)\,dx - B\frac{d\phi}{dx}(L) + A\frac{d\phi}{dx}(0), \\
&= -\frac{L\lambda_n}{2}b_n(t) - B\frac{d\phi}{dx}(L) + A\frac{d\phi}{dx}(0).
\end{aligned}$$

Since $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, so

$$\frac{d\phi_n}{dx}(x) = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right).$$

It follows that

$$\begin{aligned}
b_n'(t) + k\lambda_n b_n(t) &= \frac{2k}{L}\left[A\frac{d\phi}{dx}(0) - B\frac{d\phi}{dx}(L)\right], \\
b_n'(t) + k\frac{n^2\pi^2}{L^2}b_n(t) &= \frac{2kn\pi}{L^2}\left[A - B\cos(n\pi)\right],
\end{aligned}$$

which is a linear nonhomogeneous ODE in $b_n(t)$. The variation of parameters formula gives the solution:

$$\begin{aligned} b_n(t) &= e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left[b(0) + \frac{2kn\pi}{L^2}\left[A - B(-1)^2\right]\int_0^t e^{k\left(\frac{n\pi}{L}\right)^2 s} ds\right], \\ &= e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left[b(0) + \frac{2}{n\pi}\left[A - B(-1)^2\right]\left(e^{k\left(\frac{n\pi}{L}\right)^2 t} - 1\right)\right]. \end{aligned}$$

The initial condition gives:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n(0)\sin\left(\frac{n\pi x}{L}\right),$$

which gives the initial values for the ODE in $b_n(t)$:

$$b_n(0) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Thus, the solution satisfies:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi x}{L}\right),$$

where $b_n(t)$ and $b_n(0)$ are given above.

9.2.1.c. (20 pts) Consider the nonhomogeneous heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t),$$

with the following IC and homogeneous BCs

$$u(x,0) = g(x) \qquad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0.$$

The related SL problem is $\phi'' + \lambda\phi = 0$ with BCs $\phi'(0) = 0$ and $\phi'(L) = 0$, which we have solved before (Neumann BCs) with eigenvalues and eigenfunctions:

$$\lambda_0 = 0 \quad \text{with} \quad \phi_0(x) = 1 \qquad \text{and} \qquad \lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{with} \quad \phi_n(x) = \cos\left(\tfrac{n\pi x}{L}\right), \quad n = 1, 2, ...$$

Using eigenfunction expansions gives:

$$u(x,t) = b_0(t) + \sum_{n=1}^{\infty} b_n(t)\phi_n(x).$$

Term by term differentiation in t is justified, and because of the homogeneous BCs we can term by term differentiate twice in x, so

$$\frac{\partial u}{\partial t} = b_0{}'(t) + \sum_{n=1}^{\infty} b_n{}'(t)\phi_n(x) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} b_n(t)\phi_n{}''(x).$$

We assume that $Q(x,t)$ has an eigenfunction expansion with:

$$Q(x,t) = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x),$$

where

$$q_0(t) = \frac{1}{L}\int_0^L Q(x,t)dx \quad \text{and} \quad q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\phi_n(x)dx.$$

The information above is substituted into the PDE to give:

$$\begin{aligned} {b_0}'(t) + \sum_{n=1}^{\infty} {b_n}'(t)\phi_n(x) &= k\sum_{n=1}^{\infty} b_n(t){\phi_n}''(x) + q_0(t) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x), \\ &= k\sum_{n=1}^{\infty} b_n(t)\left(-\lambda_n\phi_n(x)\right) + q_0(t) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \end{aligned}$$

Using orthogonality we have the following ODEs in the time dependent Fourier coefficients:

$$\begin{aligned} {b_0}'(t) &= q_0(t), \\ {b_n}'(t) + \lambda_n b_n(t) &= q_n(t). \end{aligned}$$

The IC gives:

$$u(x,0) = g(x) = b_0(0) + \sum_{n=1}^{\infty} b_n(0)\cos\left(\tfrac{n\pi x}{L}\right),$$

which has the Fourier coefficients:

$$b_0(0) = \frac{1}{L}\int_0^L g(x)dx \qquad \text{and} \qquad b_n(0) = \frac{2}{L}\int_0^L g(x)\cos\left(\tfrac{n\pi x}{L}\right)dx, \quad n = 1, 2, ...$$

The solution to the ODEs for $b_n(t)$:

$$b_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(b_n(0) + \int_0^t q_n(s)e^{k\left(\frac{n\pi}{L}\right)^2 s}ds\right), \qquad n = 0, 1, 2, ...$$

The solution is given by:

$$u(x,t) = \left[b_0(0) + \int_0^t q_0(s)ds\right] + \sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(b_n(0) + \int_0^t q_n(s)e^{k\left(\frac{n\pi}{L}\right)^2 s}ds\right)\cos\left(\tfrac{n\pi x}{L}\right).$$

Thus,

$$\begin{aligned} u(x,t) &= \frac{1}{L}\int_0^L g(\xi)d\xi + \sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\frac{2}{L}\int_0^L g(\xi)\cos\left(\frac{n\pi\xi}{L}\right)d\xi\cos\left(\tfrac{n\pi x}{L}\right) \\ &\quad + \frac{1}{L}\int_0^t\int_0^L Q(\xi,s)d\xi\, ds + \sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\frac{2}{L}\int_0^t\int_0^L Q(\xi,s)\cos\left(\frac{n\pi\xi}{L}\right)d\xi\, e^{k\left(\frac{n\pi}{L}\right)^2 s}ds\cos\left(\tfrac{n\pi x}{L}\right). \end{aligned}$$

We exchange the integral and sum to give:

$$\begin{aligned} u(x,t) &= \int_0^L g(\xi)\left[\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi\xi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right]d\xi \\ &+\int_0^L\int_0^t Q(\xi,s)\left[\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}\cos\left(\frac{n\pi\xi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right]ds\,d\xi. \end{aligned}$$

In Green's function notation we have:

$$u(x,t)=\int_0^L g(\xi)G(x,t;\xi,0)d\xi+\int_0^L\int_0^t Q(\xi,s)G(x,t;\xi,s)ds\,d\xi,$$

where $G(x,t;\xi,s)=\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}\cos\left(\frac{n\pi\xi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$.

d. Consider the nonhomogeneous heat equation:

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}+Q(x,t),$$

with the following IC and nonhomogeneous BCs

$$u(x,0)=g(x)\qquad\text{with}\quad \frac{\partial u}{\partial x}(0,t)=A(t)\quad\text{and}\quad\frac{\partial u}{\partial x}(L,t)=B(t).$$

The related SL problem is $\phi''+\lambda\phi=0$ with BCs $\phi'(0)=0$ and $\phi'(L)=0$, which we have solved before (Neumann BCs) with eigenvalues and eigenfunctions:

$$\lambda_0=0\quad\text{with}\quad\phi_0(x)=1\qquad\text{and}\qquad\lambda_n=\frac{n^2\pi^2}{L^2}\quad\text{with}\quad\phi_n(x)=\cos\left(\frac{n\pi x}{L}\right),\quad n=1,2,...$$

We seek solutions of the form:

$$u(x,t)=b_0(t)+\sum_{n=1}^{\infty}b_n(t)\phi_n(x)\qquad\text{with}\qquad b_n(t)=\frac{2}{L}\int_0^L u(x,t)\phi_n(x)dx.$$

Differentiating the coefficient formula gives:

$$b_n{}'(t)=\frac{2}{L}\int_0^L\frac{\partial u}{\partial t}\phi_n(x)dx=\frac{2}{L}\int_0^L\left(k\frac{\partial^2 u}{\partial x^2}+Q(x,t)\right)\phi_n(x)dx.$$

We apply Green's formula to the integral with u_{xx}:

$$\int_0^L\frac{\partial^2 u}{\partial x^2}\phi_n(x)dx=\int_0^L u(x,t)\phi_n{}''(x)dx+\left[\frac{\partial u}{\partial x}\phi_n-u\phi'\right]\Big|_0^L.$$

The BCs and eigenvalue problem imply that

$$\int_0^L u(x,t)\phi_n{}''(x)dx+\left[\frac{\partial u}{\partial x}\phi_n-u\phi'\right]\Big|_0^L=-\lambda_n\int_0^L u(x,t)\phi_n(x)dx+B(t)\cos(n\pi)-A(t)\cos(0).$$

However, $\int_0^L u(x,t)\phi_n(x)dx=\frac{L}{2}b_n(t)$, so

$$b_n{}'(t)+k\left(\frac{n\pi}{L}\right)^2 b_n(t)=\frac{2k}{L}[B(t)\cos(n\pi)-A(t)]+\frac{2}{L}\int_0^L Q(x,t)\phi_n(x)dx.$$

The IC gives:

$$u(x,0) = g(x) = b_0(0) + \sum_{n=1}^{\infty} b_n(0)\cos\left(\tfrac{n\pi x}{L}\right),$$

which has the Fourier coefficients:

$$b_0(0) = \frac{1}{L}\int_0^L g(\xi)d\xi \qquad \text{and} \qquad b_n(0) = \frac{2}{L}\int_0^L g(\xi)\cos\left(\tfrac{n\pi\xi}{L}\right)d\xi, \quad n = 1,2,...$$

The solution to the ODEs for $b_n(t)$:

$$b_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(b_n(0) + \frac{2}{L}\int_0^t \left[kB(s)\cos(n\pi) - kA(s) + \int_0^L Q(\xi,s)\phi_n(\xi)d\xi\right] e^{k\left(\frac{n\pi}{L}\right)^2 s}ds\right),$$

for $n = 1,2,...$, and

$$b_0(t) = b_0(0) + \frac{1}{L}\int_0^t \left[kB(s) - kA(s) + \int_0^L Q(\xi,s)d\xi\right] ds.$$

The solution is given by:

$$\begin{aligned} u(x,t) \;=\; & \frac{1}{L}\int_0^L g(\xi)d\xi + \frac{1}{L}\int_0^t \left[kB(s) - kA(s) + \int_0^L Q(\xi,s)d\xi\right] ds + \\ & \sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\left(\frac{2}{L}\int_0^L g(\xi)\cos\left(\tfrac{n\pi\xi}{L}\right)d\xi + \right. \\ & \left.\frac{2}{L}\int_0^t \left[kB(s)\cos(n\pi) - kA(s) + \int_0^L Q(\xi,s)\phi_n(\xi)d\xi\right] e^{k\left(\frac{n\pi}{L}\right)^2 s}ds\cos\left(\tfrac{n\pi x}{L}\right)\right). \end{aligned}$$

We exchange the integral and sum to give:

$$\begin{aligned} u(x,t) \;=\; & \int_0^L g(\xi)\left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\tfrac{n\pi\xi}{L}\right)\cos\left(\tfrac{n\pi x}{L}\right)\right]d\xi \\ & + \int_0^L\int_0^t Q(\xi,s)\left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}\cos\left(\tfrac{n\pi\xi}{L}\right)\cos\left(\tfrac{n\pi x}{L}\right)\right]ds\,d\xi \\ & + k\int_0^t B(s)\left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}\cos(n\pi)\cos\left(\tfrac{n\pi x}{L}\right)\right]ds \\ & - k\int_0^t A(s)\left[\frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}\cos\left(\tfrac{n\pi x}{L}\right)\right]ds \end{aligned}$$

In Green's function notation we have:

$$\begin{aligned} u(x,t) \;=\; & \int_0^L g(\xi)G(x,t;\xi,0)d\xi + \int_0^L\int_0^t Q(\xi,s)G(x,t;\xi,s)ds\,d\xi \\ & + k\int_0^t B(s)G(x,t;L,s) - k\int_0^t A(s)G(x,t;0,s), \end{aligned}$$

where $G(x,t;\xi,s) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 (t-s)}\cos\left(\frac{n\pi\xi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$.

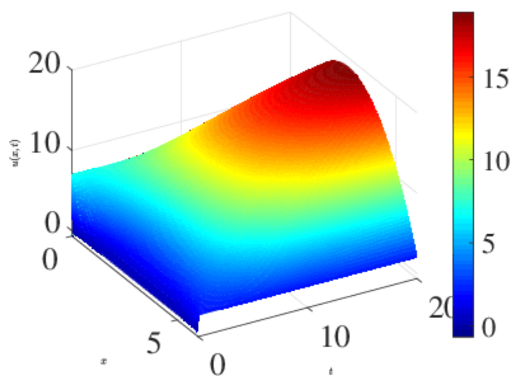
WW8.2.1 (10pts) We consider the nonhomogeneous heat equation:

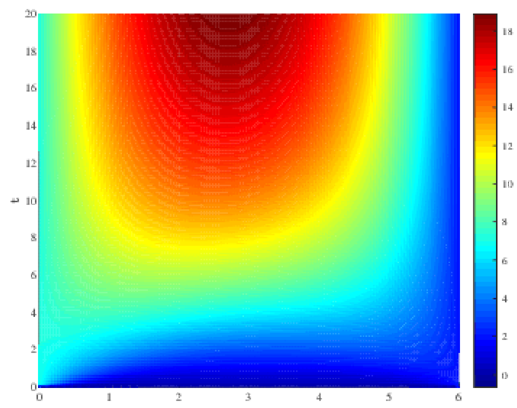
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q, \qquad 0 < x < L, \quad t > 0,$$

with nonhomogeneous boundary conditions and initially zero temperature, so

$$u(0,t) = A, \qquad u(L,t) = B, \qquad \text{and} \qquad u(x,0) = 0.$$

WeBWorK randomizes the values k, q, A, and B and checks to see that the correct *equilibrium solution* is found and that once subtracted from the nonhomogeneous problem above, the resulting homogeneous problem is solved with our standard techniques. The graph is representative of the type of graph all solutions have. The plot begins with the temperature approximating 0 with the hundred terms. Immediately, the boundaries at $x = 0$ and $x = L$ jump to two nonzero values and stay at that level for the rest of the time (*Dirichlet BCs*. The equilibrium solution is a quadratic, as heat is generated throughout the bar and diffuses. The result is a surface that shows a growing solution approaching this quadratic equilibrium curve in t.





The MatLab program producing the 50-term graph is presented below.

```
%format compact;
NptsX = 151;            % number of x pts
NptsT = 151;            % number of t pts
Nf = 100;                % number of Fourier terms
x = linspace(0,6,NptsX);
t = linspace(0,20,NptsT);
[X,T] = meshgrid(x,t);

fs=8;
figure(101)
clf

b = zeros(1,Nf);
U = zeros(NptsT,NptsX);
Ue = -15/8*X.^2 + 125/12*X + 7;
kh = 0.4*pi^2/36;
for n=1:Nf
    b(n) = (4*n^2*pi^2*cos(n*pi) - 14*n^2*pi^2 ...
        + 270*cos(n*pi) - 270)/(n^3*pi^3);  % Fourier coefficients
    Un=b(n)*sin(n*pi*X/6).*exp(-kh*n^2*T);  % Temperature(n)
    U=U+Un;
```

```
end
U = U + Ue;
set(gca,'FontSize',[fs]);
surf(X,T,U);
shading interp
colormap(jet)
fontlabs = 'Times New Roman';
xlabel('$x$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
ylabel('$t$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
%axis tight
colorbar
view([60 35])
print -depsc WW8_2a.eps

figure(102)
clf

set(gca,'FontSize',[fs]);
surf(X,T,U);
shading interp
colormap(jet)
view([0 90])              %create 2D color map of temperature
xlabel('x','Fontsize',fs);
ylabel('t','Fontsize',fs);
zlabel('u(x,t)','Fontsize',fs);axis tight
colorbar
set(gca,'FontSize',[fs]);

print -depsc WW8_2b.eps
```