1. a. Consider the Sturm-Liouville problem for $x \in (0,3)$:

$$u'' + \lambda u = 0, \quad u'(0) = 0, \quad u(3) = 0.$$

We consider 3 cases:

Case (i): If $\lambda = 0$, then $u(x) = c_1 x + c_2$. One BC gives $u'(0) = c_1 = 0$. The other BC gives $u(3) = c_2 = 0$, which gives only the trivial solution.

Case (ii): If $\lambda = -\alpha^2 < 0$, then $u(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$. One BC gives $u'(0) = c_2\alpha = 0$ or $c_2 = 0$. The other BC gives $u(3) = c_1 \cosh(3\alpha) = 0$ or $c_1 = 0$, which gives only the trivial solution.

Case (iii): If $\lambda = \alpha^2 > 0$, then $u(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. One BC gives $u'(0) = c_2\alpha = 0$ or $c_2 = 0$. The other BC gives $u(3) = c_1 \cos(3\alpha) = 0$, so for non-trivial solutions,
$\alpha_n = \frac{(2n-1)\pi}{6}$, $n = 1, 2, ...$
It follows that the eigenvalues and corresponding eigenfunctions are:

$$\lambda_n = \frac{(2n-1)^2\pi^2}{36} \qquad \text{and} \qquad u_n(x) = \cos\left(\frac{(2n-1)\pi x}{6}\right), \quad n = 1, 2, ...$$

b. The eigenfunctions from Part a form a complete orthogonal set, so we represent the function:

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 2, & 1 \le x \le 3. \end{cases} \qquad \text{by} \qquad f(x) \sim \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n-1)\pi x}{6}\right).$$

Orthogonality gives:

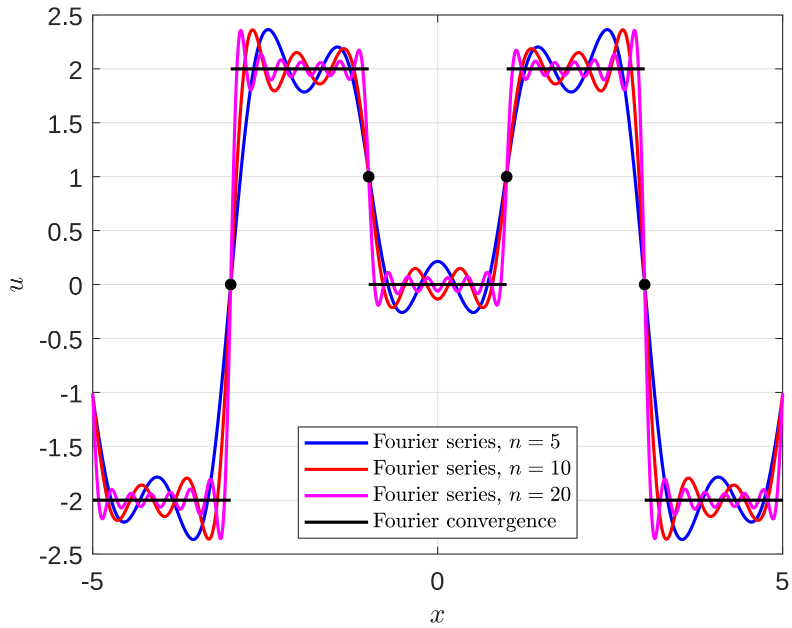
$$\int_0^3 f(x)\cos\left(\frac{(2n-1)\pi x}{6}\right) dx = \int_1^3 2\cos\left(\frac{(2n-1)\pi x}{6}\right) dx = A_n \int_0^3 \cos^2\left(\frac{(2n-1)\pi x}{6}\right) dx$$

It follows that

$$\begin{aligned} A_n &= \frac{4}{3}\int_1^3 \cos\left(\frac{(2n-1)\pi x}{6}\right) dx = \frac{8}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{6}\right)\Bigg|_1^3 \\ &= \frac{8}{(2n-1)\pi}\left[\sin\left(\frac{(2n-1)\pi x}{2}\right) - \sin\left(\frac{(2n-1)\pi}{6}\right)\right] \\ &= \frac{8}{(2n-1)\pi}\left[(-1)^{n+1} + \cos\left(\frac{\pi}{3}(n+1)\right)\right] \end{aligned}$$

c. This Fourier series converges to **2** at $x = 2$ (pt. of cont.). It converges to **1** at $x = 1$ (midpoint of jump). It converges to **2** at $x = -\frac{5}{2}$ (even extension).

d. Below we show the graph of the approximation of $f(x)$ using $n = 5, 10, 20$ terms in the Fourier series for $x \in [-5, 5]$. In black, the graph shows the points of convergence of the Fourier series on this interval.



The absolute error between the 20 term Fourier series and the $f(x)$ at various values of x satisfies:

x	Fourier series	Absolue error
0.1	0.03531	0.03531
0.95	0.3363	0.3363
2	2.05992	0.05992
2.75	1.8922	0.1078

The maximum value of the function with 20 terms is 2.35868, which occurs at $x = 2.85367$, so the maximum absolute error is 0.35868, which is roughly 9% of the jump.

```
% Periodic Fourier cosine series TH1 s23

NptsX=1001;            % number of x pts
x = linspace(-5,5,NptsX);
f1 = zeros(1,NptsX);
f2 = zeros(1,NptsX);
f3 = zeros(1,NptsX);
for n=1:5
    b(n)=8/((2*n-1)*pi)*(sin((n-1/2)*pi) - sin((n-1/2)*pi/3));
    fn=b(n)*cos((n-1/2)*pi*x/3);   % Fourier function(n)
    f1=f1+fn;
end
for n=1:10
    b(n)=8/((2*n-1)*pi)*(sin((n-1/2)*pi) - sin((n-1/2)*pi/3));
    fn=b(n)*cos((n-1/2)*pi*x/3);   % Fourier function(n)
    f2=f2+fn;
end
for n=1:20
    b(n)=8/((2*n-1)*pi)*(sin((n-1/2)*pi) - sin((n-1/2)*pi/3));
    fn=b(n)*cos((n-1/2)*pi*x/3);   % Fourier function(n)
    f3=f3+fn;
end
```

```
plot(x,f1,'b-','LineWidth',1.5);
hold on
plot(x,f2,'r-','LineWidth',1.5);
plot(x,f3,'m-','LineWidth',1.5);
plot([-1 1],[0,0],'k-','LineWidth',1.5);
plot([1 3],[2 2],'k-','LineWidth',1.5);
plot([-3 -1],[2 2],'k-','LineWidth',1.5);
plot([-5 -3],[-2 -2],'k-','LineWidth',1.5);
plot([3 5],[-2 -2],'k-','LineWidth',1.5);
plot([-1 1],[1 1],'ko','MarkerSize',5,'MarkerFaceColor','k');
plot([-3 3],[0 0],'ko','MarkerSize',5,'MarkerFaceColor','k');
grid;
h = legend('Fourier series, $n = 5$', 'Fourier series, $n = 10$',...
     'Fourier series, $n = 20$','Fourier convergence', 'Location','south');
set(h,'Interpreter','latex')
h.FontSize = 10;
xlim([-5,5]);
ylim([-2.5 2.5]);
xlabel('$x$','FontSize',12,'interpreter','latex');
ylabel('$u$','FontSize',12,'interpreter','latex');
set(gca,'FontSize',12);         % Axis tick font size
print -depsc th1_1d_s23.eps
```

2. a. The string problem satisfies the nonhomogeneous partial differential equation:

$$u_{tt} + 2ku_t = c^2 u_{xx} - g, \qquad t > 0 \quad \text{and} \quad 0 < x < 1,$$

with $k > 0$ ($k \ll c\pi$) and $g > 0$.The boundary conditions are $u(0,t) = 0$ and $u(1,t) = 0$. The equilibrium solution satisfies:

$$c^2 u_E'' - g = 0, \qquad u_E(0) = 0 \quad \text{and} \quad u_E(1) = 0.$$

Since $u_E'' = \frac{g}{c^2}$, we integrate twice to give

$$u_E(x) = \frac{g}{2c^2}x^2 + c_1 x + c_2.$$

The boundary conditions give $u_E(0) = 0 = c_2$ and $u_E(1) = 0 = \frac{g}{2c^2} + c_1$ or $c_1 = -\frac{g}{2c^2}$. It follows that the *equilibrium solution* is

$$u_E(x) = \frac{g}{2c^2}(x^2 - x).$$

b. We let $w(x,t) = u(x,t) - u_E(x)$, then $w_t = u_t$, $w_{tt} = u_{tt}$, and $w_{xx} = u_{xx} - u_E''$. However, $u_E'' = \frac{g}{2c^2}$, so when substituted into the string problem, we have

$$w_{tt} + 2kw_t = c^2\left(w_{xx} + \frac{g}{c^2}\right) - g = c^2 w_{xx}.$$

This gives a damped linear homogeneous wave equation in w with the homogeneous boundary conditions:

$$w(0,t) = 0 \qquad \text{and} \qquad w(1,t) = 0.$$

The initial conditions are:

$$w(x,0) = u(x,0) - u_E(x) = 0 \qquad \text{and} \qquad w_t(x,0) = 1.$$

We solve the equation in w using separation of variables, so $w(x,t) = \phi(x)h(t)$ and

$$\phi h'' + k\phi h' = c^2\phi'' h \qquad \text{or} \qquad \frac{h'' + 2kh'}{c^2 h} = \frac{\phi''}{\phi} = -\lambda.$$

The Sturm-Liouville problem is

$$\phi'' + \lambda\phi = 0 \qquad \text{with} \qquad \phi(0) = 0 \quad \text{and} \quad \phi(1) = 0.$$

This is a standard SL problem with Dirichlet BCs, giving the eigenvalues and associated eigenvectors:

$$\lambda_n = n^2\pi^2 \qquad \text{and} \qquad \phi_n(x) = \sin(n\pi x).$$

The t-equation is

$$h_n'' + 2kh_n' + (n^2\pi^2c^2)h_n = 0,$$

which has the characteristic equation:

$$r^2 + 2kr + n^2\pi^2c^2 = 0, \qquad \text{so} \quad r = -k \pm i\omega_n,$$

where $\omega_n^2 = n^2\pi^2c^2 - k^2 > 0$. This gives the general solution:

$$h_n(t) = e^{-kt}[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)].$$

Since the initial position is zero, $h_n(0) = 0$ and $A_n = 0$. We now apply the Superposition Principle and obtain:

$$w(x,t) = \sum_{n=0}^{\infty} B_n e^{-kt}\sin(\omega_n t)\sin(n\pi x).$$

The velocity is:

$$w_t(x,t) = \sum_{n=0}^{\infty} B_n e^{-kt}[-k\sin(\omega_n t) + \omega_n\cos(\omega_n t)]\sin(n\pi x),$$

so the initial velocity is

$$w_t(x,0) = 1 = \sum_{n=0}^{\infty} B_n\omega_n\sin(n\pi x).$$

Orthogonality of the eigenfunctions gives us the Fourier coefficients:

$$B_n = \frac{2}{\omega_n}\int_0^1 \sin(n\pi x)dx = \frac{2}{n\pi\omega_n}[1 - \cos(n\pi)] = \frac{2}{n\pi\omega_n}[1 - (-1)^n].$$

It follows that

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-kt}\sin(\omega_n t)\sin(n\pi x) + \frac{g}{2c^2}(x^2 - x),$$

where

$$B_n = \frac{2}{n\pi\omega_n}[1 - (-1)^n] \qquad \text{and} \qquad \omega_n = \sqrt{n^2\pi^2c^2 - k^2}.$$

With the exponential decay in the Fourier series, we have

$$\lim_{t\to\infty} u(x,t) = u_E(x) = \frac{g}{2c^2}(x^2 - x).$$

3. Given the heat equation:

$$\frac{\partial u}{\partial t} = k\nabla^2 u, \quad 0 < x < 2, \quad 0 < y < 3, \quad t > 0.$$

with boundary conditions:

$$\frac{\partial u}{\partial x}(0, y, t) = A(3-y), \quad \frac{\partial u}{\partial x}(2, y, t) = y^2, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \text{and} \quad \frac{\partial u}{\partial y}(x, 3, t) = 0,$$

we begin with the solvability condition:

$$\oint \nabla u \cdot \mathbf{n}\, ds = 0,$$

which for this problem becomes:

$$\int_0^2 u_y(x, 0, t)dx + \int_0^3 u_x(2, y, t)dx + \int_2^0 u_y(x, 3, t)dx + \int_3^0 u_x(0, y, t)dx = 0.$$

Inserting the B.C.'s, we find that:

$$\begin{aligned} \int_0^2 0\, dx + \int_0^3 y^2\, dy + \int_2^0 0\, dx + \int_3^0 A(3-y)dy &= \left.\frac{y^3}{3}\right|_0^3 - \left.\left(3Ay - \frac{Ay^2}{2}\right)\right|_0^3, \\ &= 9 - \left(9A - \frac{9A}{2}\right) = 9 - \frac{9A}{2} = 0. \end{aligned}$$

This implies that $A = 2$.

For the steady-state problem we split the original problem into 2 problems, $\nabla^2 u_1 = 0$ with $\frac{\partial u_1}{\partial x}(0, y) = A(3-y)$ and $\nabla^2 u_2 = 0$ with $\frac{\partial u_2}{\partial x}(2, y) = y^2$ and all other B.C.'s for each problem are homogeneous (Neumann). As usual we start with separation of variables, $u_1(x, y) = h(x)\phi(y)$, so $h''\phi + h\phi'' = 0$ or

$$\frac{h''}{h} = -\frac{\phi''}{\phi} = \lambda.$$

The SL problem is $\phi'' + \lambda\phi = 0$ with $\phi'(0) = 0 = \phi'(3)$. We have solved this eigenvalue problem before, and we obtained the eigenvalues and corresponding eigenfunctions:

$$\lambda_0 = 0 \quad \text{with} \quad \phi_0(y) = 1, \qquad \text{and} \qquad \lambda_n = \frac{n^2\pi^2}{9} \quad \text{with} \quad \phi_n(y) = \cos\left(\frac{n\pi y}{3}\right).$$

The x-equation satisfies $h'' - \frac{n^2\pi^2}{9}h = 0$ with $h'(2) = 0$, so we can write:

$$h_n(x) = c_1 \cosh\left(\frac{n\pi}{3}(2-x)\right) + c_2 \sinh\left(\frac{n\pi}{3}(2-x)\right).$$

The B.C. implies that $c_2 = 0$. For $\lambda_0 = 0$, $h_0(x) = c_1 + c_2 x$ with $h_0{}'(2) = 0$, implying that $c_2 = 0$. Combining these results with the Superposition Principle gives:

$$u_1(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi y}{3}\right) \cosh\left(\frac{n\pi}{3}(2-x)\right).$$

From the nonhomogeneous B.C., we find the Fourier coefficients. We see that for $A = 2$:

$$\frac{\partial u_1}{\partial x}(0, y) = 2(3 - y) = -\sum_{n=1}^{\infty} A_n \frac{n\pi}{3} \cos\left(\frac{n\pi y}{3}\right) \sinh\left(\frac{2n\pi}{3}\right).$$

Taking advantage of orthogonality, we obtain:

$$-A_n \frac{n\pi}{3} \sinh\left(\frac{2n\pi}{3}\right) \int_0^3 \cos^2\left(\frac{n\pi y}{3}\right) dy = 2 \int_0^3 (3 - y) \cos\left(\frac{n\pi y}{3}\right) dy.$$

With Maple doing the integral on the right hand side and some algebra, we have:

$$A_n = \frac{36}{n^3 \pi^3 \sinh\left(\frac{2n\pi}{3}\right)} \left((-1)^n - 1\right).$$

We now let $u_2(x, y) = h(x)\phi(y)$, and the SL problem is the same as for $u_1(x, y)$. We solve $h'' - \frac{n^2\pi^2}{9} h = 0$ with $h'(0) = 0$, so we can write:

$$h_n(x) = c_1 \cosh\left(\frac{n\pi x}{3}\right) + c_2 \sinh\left(\frac{n\pi x}{3}\right).$$

The B.C. implies that $c_2 = 0$. For $\lambda_0 = 0$, $h_0(x) = c_1 + c_2 x$ with $h_0{}'(2) = 0$, implying that $c_2 = 0$. Combining these results with the Superposition Principle gives:

$$u_2(x, y) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi y}{3}\right) \cosh\left(\frac{n\pi x}{3}\right).$$

From the nonhomogeneous B.C., we find the Fourier coefficients. We see that:

$$\frac{\partial u_2}{\partial x}(2, y) = y^2 = \sum_{n=1}^{\infty} B_n \frac{n\pi}{3} \cos\left(\frac{n\pi y}{3}\right) \sinh\left(\frac{2n\pi}{3}\right).$$

Taking advantage of orthogonality, we obtain:

$$B_n \frac{n\pi}{3} \sinh\left(\frac{2n\pi}{3}\right) \frac{3}{2} = \int_0^3 y^2 \cos\left(\frac{n\pi y}{3}\right) dy.$$

With Maple doing the integral on the right hand side and some algebra, we have:

$$B_n = \frac{108(-1)^n}{n^3 \pi^3 \sinh\left(\frac{2n\pi}{3}\right)}.$$

It remains to show the constants A_0 and B_0, which depend on the initial conditions. We have:

$$\begin{aligned}
\int_0^3 \int_0^2 (A_0 + B_0) dx\, dy &= (A_0 + B_0)6, \\
\int_0^3 \int_0^2 u(x, y) dx\, dy &= \int_0^3 \int_0^2 x(3 - y) dx\, dy = \left. \left. \frac{x^2(3y - y^2/2)}{2} \right|_0^2 \right|_0^3 = 9.
\end{aligned}$$

It follows that $A_0 + B_0 = \frac{3}{2}$. Combining all these results gives:

$$u(x,y) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{36}{n^3\pi^3 \sinh\left(\frac{2n\pi}{3}\right)} \left(((-1)^n - 1)\cosh\left(\frac{n\pi}{3}(2-x)\right) + 3(-1)^n \cosh\left(\frac{n\pi x}{3}\right)\right) \cos\left(\frac{n\pi y}{3}\right).$$

4. a. The steady-state temperature distribution of the semi-circular region is given by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

which has the B.C.'s $\frac{\partial u}{\partial \theta}(r,0) = 0$, $u(r,\pi) = 0$, and $u(2,\theta) = \pi - \theta$. There is the implicit B.C. that $\lim_{r\to 0} |u(r,\theta)|$ is bounded. Let $u(r,\theta) = h(r)\phi(\theta)$, then Laplace's equation gives:

$$\frac{\phi}{r}\frac{d}{dr}\left(r\frac{dh}{dr}\right) + \frac{h\phi''}{r^2} = 0 \qquad \text{or} \qquad \frac{r\,(rh')'}{h} = -\frac{\phi''}{\phi} = \lambda.$$

The SL problem is:

$$\phi'' + \lambda\phi = 0, \qquad \phi'(0) = 0 \quad \text{and} \quad \phi(\pi) = 0.$$

If $\lambda = 0$, then $\phi(\theta) = c_1\theta + c_2$. The B.C. $\phi'(0) = 0 = c_1$. The B.C. $\phi(\pi) = 0 = c_2$, so this gives the trivial solution, which implies that $\lambda = 0$ is not an eigenvalue.

If $\lambda = -\alpha^2 < 0$, then $\phi(\theta) = c_1\cosh(\alpha\theta) + c_2\sinh(\alpha\theta)$. The B.C. $\phi'(0) = 0 = c_2\alpha$. The B.C. $\phi(\pi) = 0 = c_1\cosh(\alpha\pi)$. Since $\cosh(\alpha\pi) > 0$, $c_1 = 0$, which again gives only the trivial solution, so $\lambda < 0$ does not produce any eigenvalues.

If $\lambda = \alpha^2 < 0$, then $\phi(\theta) = c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta)$. The B.C. $\phi'(0) = 0 = c_2\alpha$. The B.C. $\phi(\pi) = 0 = c_1\cos(\alpha\pi)$. For a nontrivial solution we need $\cos(\alpha\pi) = 0$. It follows that $\alpha_n = \frac{2n-1}{2}$. Thus, the eigenvalues and corresponding eigenfunctions are:

$$\lambda_n = \frac{(2n-1)^2}{4} \qquad \text{with} \qquad \phi_n(\theta) = \cos\left(\frac{(2n-1)\theta}{2}\right).$$

The r-equation becomes:

$$r\left(rh'\right)' - \frac{(2n-1)^2}{4}h = 0 \qquad \text{or} \qquad r^2h'' + rh' - \frac{(2n-1)^2}{4}h = 0.$$

Try a solution $h(r) = r^z$, and then this Cauchy-Euler equation has the auxiliary equation:

$$z^2 - \frac{(2n-1)^2}{4} = 0, \qquad \text{so} \qquad z = \pm\frac{(2n-1)}{2}.$$

It follows that the general solution is:

$$h_n(r) = c_1 r^{(2n-1)/2} + c_2 r^{-(2n-1)/2}.$$

The boundedness condition as $r \to 0$ implies that $c_2 = 0$.

The Superposition Principle gives:

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n r^{(2n-1)/2} \cos\left(\frac{(2n-1)\theta}{2}\right).$$

It remains to satisfy the remaining B.C., $u(2,\theta) = \pi - \theta$. Thus, we require

$$u(2,\theta) = \sum_{n=1}^{\infty} a_n 2^{(2n-1)/2} \cos\left(\frac{(2n-1)\theta}{2}\right) = \pi - \theta.$$

By the orthogonality of the eigenfunctions, we have

$$a_n 2^{(2n-1)/2}\left(\frac{\pi}{2}\right) = \int_0^{\pi} (\pi - \theta)\cos\left(\frac{(2n-1)\theta}{2}\right) d\theta.$$

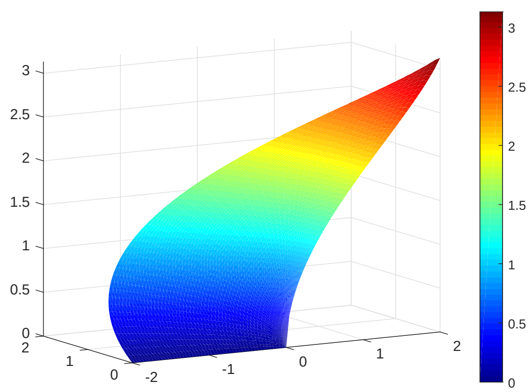
The integral on the right is solved by Maple, and the Fourier coefficient is given by:

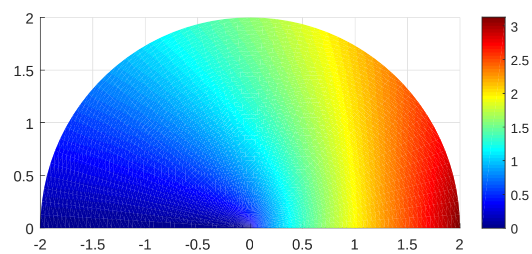
$$a_n = \frac{8(2^{-(2n-1)/2})}{\pi(4n^2 - 4n + 1)},$$

so our solution is given by:

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{8(2^{-(2n-1)/2})r^{(2n-1)/2}}{\pi(4n^2 - 4n + 1)} \cos\left(\frac{(2n-1)\theta}{2}\right).$$

b. Below is a colored heat map displaying the steady-state temperature distribution in this region. You must include your program.





```
%semicircular heat distribution

NptsR = 51;
NptsT = 51;
Nf = 50;

r=linspace(0,2,NptsR);
t=linspace(0,pi,NptsT);
[R,T]=meshgrid(r,t);
X=R.*cos(T);
Y=R.*sin(T);

fs=8;
figure(101)
clf

b=zeros(1,Nf);
```

```
U=zeros(NptsR,NptsT);

for n=1:Nf
    b(n)=(8*2^(-(2*n-1)/2))/(pi*(4*n^2-4*n+1));  % Fourier coefficients
    Un=b(n)*R.^((2*n-1)/2).*cos((2*n-1)*T/2);  % Temperature(n)
    U=U+Un;
end

colormap jet;
surf(X,Y,U)
shading interp;
axis equal;
colorbar
view(-30,10);

print -depsc ss_semis19_heata.eps        % Color

figure(102)

surf(X,Y,U)
colormap jet;
shading interp;
axis equal;
colorbar
view(0,90);

print -depsc ss_semis19_heatb.eps
```

5. a. We are given the ODE:

$$\phi'' - 0.4\phi' + \lambda\phi = 0, \qquad \phi(0) = 0, \quad \phi(8) = 0.$$

The Sturm-Liouville eigenvalue problem satisfies:

$$[p\phi']' + q\phi + \lambda\sigma\phi = 0 \qquad \text{or} \qquad p\phi'' + p'\phi' + q\phi + \lambda\sigma\phi = 0.$$

We convert the ODE above into a Sturm-Liouville eigenvalue problem by choosing a function $H(x)$ to multiply the equation above:

$$H\phi'' - 0.4H\phi' + \lambda H\phi = 0.$$

It follows that $p(x) = H(x)$ and $p' = H' = -0.4H$. Thus,

$$p(x) = H(x) = e^{-0.4x}, \quad q(x) = 0, \quad \text{and} \quad \sigma(x) = e^{-0.4x}.$$

The SL problem satisfies:

$$\frac{d}{dx}\left(e^{-0.4x}\frac{d\phi}{dx}\right) + \lambda e^{-0.4x}\phi = 0.$$

The characteristic polynomial is $r^2 - 0.4r + \lambda = (r-0.2)^2 + \lambda - 0.04 = 0$. The Rayleigh Quotient readily shows that $\lambda \geq 0$, as

$$\lambda = \frac{-p(x)\phi(x)\phi'(x)\big|_0^8 + \int_0^8 \left(p(x)(\phi'(x))^2 - q(x)(\phi(x))^2\right) dx}{\int_0^8 (\phi(x))^2\sigma(x)dx} \geq 0,$$

where $\phi(0) = \phi(8) = 0$, $p(x) = \sigma(x) > 0$, $q(x) = 0$, $\phi^2(x) > 0$, and $(\phi'(x))^2 \geq 0$.

If $\lambda = 0$, then $\phi'' - 0.4\phi' = 0$, which has the general solution, $\phi(x) = c_1 + c_2 e^{0.4x}$, so $\phi(0) = c_1 + c_2 = 0$. The other BC gives $c_1 + c_2 e^{3.2} = 0$, so $c_1 = c_2 = 0$, leaving only the trivial solution.

If $\lambda - 0.04 = -\alpha^2 < 0$, then $r = 0.2 \pm \alpha$, which has the solution,
$\phi(x) = e^{0.2x}(c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x))$. The B.C. $\phi(0) = c_1 = 0$. The other B.C. gives $\phi(8) = c_2 e^{1.6} \sinh(8\alpha) = 0$, which implies $c_2 = 0$, leaving only the trivial solution.

If $\lambda = 0.04$, then $r = 0.2$, which has the solution, $\phi(x) = e^{0.2x}(c_1 + c_2 x)$. The B.C. $\phi(0) = c_1 = 0$. The other B.C. gives $\phi(8) = c_2 5 e^{1.6} = 0$, which implies $c_2 = 0$, leaving only the trivial solution.

If $\lambda - 0.04 = \alpha^2 > 0$, then $r = 0.2 \pm i\alpha$, which has the solution,
$\phi(x) = e^{0.2x}(c_1 \cos(\alpha x) + c_2 \sin(\alpha x))$. The B.C. $\phi(0) = c_1 = 0$. The other B.C. gives $\phi(8) = c_2 e^{1.6} \sin(8\alpha) = 0$, which has nontrivial solutions if $\alpha_n = \frac{n\pi}{8}$. It follows that the eigenvalues and eigenfunctions are:

$$\lambda_n = 0.04 + \left(\frac{n\pi}{8}\right)^2, \qquad \text{with} \quad \phi_n(x) = e^{0.2x} \sin\left(\frac{n\pi x}{8}\right), \quad n = 1, 2, 3, ...$$

The orthogonality relation for this SL problem with eigenfunctions $\phi_m(x)$ and $\phi_n(x)$, $m \neq n$, is:

$$\int_0^8 \phi_m(x)\phi_n(x) e^{-0.4x}\, dx = 0.$$

b. A one-dimensional rod for heat conduction with convection is given by:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 0.4\frac{\partial u}{\partial x}, \quad 0 < x < 8, \quad t > 0,$$

with B.C.'s and I.C.'s:

$$u(0,t) = 0, \quad u(8,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

From separation of variables with $u(x,t) = \phi(x)h(t)$, we have:

$$\phi h' = \phi'' h - 0.4\phi' h \qquad \text{or} \qquad \frac{h'}{h} = \frac{\phi'' - 0.4\phi'}{\phi} = -\lambda.$$

This gives the t-equation with its solution:

$$h' + \lambda h = 0, \qquad \text{so} \qquad h(t) = ce^{-\lambda t}.$$

The x-equation and its B.C.'s are given by:

$$\phi'' - 0.4\phi' + \lambda\phi = 0 \qquad \text{with} \qquad \phi(0) = 0 \quad \text{and} \quad \phi(8) = 0,$$

which is the same Sturm-Liouville problem from Part a.

c. Having solved the t-equation and the SL problem in Parts a and b, we have the product solution:

$$u_n(x,t) = A_n e^{-(0.04 + n^2\pi^2/64)t} e^{0.2x} \sin\left(\frac{n\pi x}{8}\right).$$

We apply the Superposition Principle to obtain the solution:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(0.04+n^2\pi^2/64)t} e^{0.2x} \sin\left(\frac{n\pi x}{8}\right).$$

The initial condition gives:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n e^{0.2x} \sin\left(\frac{n\pi x}{8}\right).$$

We use the orthogonality relationship to find the Fourier coefficients:

$$A_n = \frac{\int_0^8 f(x)\left(e^{0.2x}\sin\left(\frac{n\pi x}{8}\right)\right)e^{-0.4x}\,dx}{\int_0^8 \left(e^{0.2x}\sin\left(\frac{n\pi x}{8}\right)\right)^2 e^{-0.4x}\,dx} = \frac{1}{4}\int_0^8 f(x)e^{-0.2x}\sin\left(\frac{n\pi x}{8}\right)\,dx.$$

6. a. The operator L is self-adjoint if

$$\int_0^6 [uL(v) - vL(u)]dx = 0.$$

Assume that the functions, $u(x)$ and $v(x)$, satisfy the B.C.'s, $u(0) = 0$, $u''(0) = 0$, $u(6) = 0$, $u''(6) = 0$, $v(0) = 0$, $v''(0) = 0$, $v(6) = 0$, and $v''(6) = 0$. We integrate by parts:

$$\int_0^6 \left(uv'''' - vu''''\right) dx = \left(uv''' - vu'''\right)\Big|_0^6 - \int_0^6 \left(u'v''' - v'u'''\right) dx.$$

Since $u(0) = u(6) = v(0) = v(6) = 0$ (fixed ends), the function evaluations vanish at the endpoints. Integration by parts a second time gives:

$$-\int_0^6 \left(u'v''' - v'u'''\right) dx = -\left(u'v'' - v'u''\right)\Big|_0^6 + \int_0^6 \left(u''v'' - v''u''\right) dx = 0,$$

because the last integral vanishes and we have the B.C.'s $u''(0) = u''(6) = v''(0) = v''(6) = 0$ (free force). This shows that L is self-adjoint.

b. From Part a, if $L(\phi_m) = \lambda_m\phi_m$ and $L(\phi_n) = \lambda_n\phi_n$, then

$$\int_0^6 \left(\phi_m L(\phi_n) - \phi_n L(\phi_m)\right) dx = \int_0^6 \left(\lambda_n\phi_m\phi_n - \lambda_m\phi_n\phi_m\right) dx = (\lambda_n - \lambda_m)\int_0^6 \phi_m\phi_n\,dx = 0,$$

which shows orthogonality of distinct eigenvalues. Suppose that λ is a complex eigenvalue with complex eigenfunction ϕ, then $L(\phi) - \lambda\phi = 0$. Taking the complex conjugate, we have

$$\overline{L(\phi) - \lambda\phi} = \bar{L}(\bar{\phi}) - \bar{\lambda}\bar{\phi} = L(\bar{\phi}) - \bar{\lambda}\bar{\phi} = 0,$$

so $\bar{\lambda}$ is an eigenvalue and $\bar{\phi}$ is its corresponding eigenfunction. However, the orthogonality relationship gives:

$$(\lambda - \bar{\lambda})\int_0^6 \phi\bar{\phi}\,dx = 0,$$

which since $\phi\bar{\phi} = |\phi|^2 > 0$ for an eigenfunction, implies that $\lambda - \bar{\lambda} = 0$, so λ is real, which contradicts the assumption of λ being complex.

From the expression related to the Rayleigh Quotient, we show that the eigenvalues are non-negative:

$$\begin{aligned}\lambda &= \frac{\int_0^6 \phi L[\phi]\,dx}{\int_0^6 \phi^2 dx} = \frac{\int_0^6 \left(\frac{d}{dx}\left(\phi\cdot\frac{d^3\phi}{dx^3} - \frac{d\phi}{dx}\cdot\frac{d^2\phi}{dx^2}\right) + \left(\frac{d^2\phi}{dx^2}\right)^2\right)dx}{\int_0^6 \phi^2 dx} \\ &= \frac{\left(\phi\cdot\frac{d^3\phi}{dx^3} - \frac{d\phi}{dx}\cdot\frac{d^2\phi}{dx^2}\right)\Big|_0^6 + \int_0^6\left(\frac{d^2\phi}{dx^2}\right)^2 dx}{\int_0^6 \phi^2 dx} \geq 0,\end{aligned}$$

since the BCs give $\left(\phi\cdot\frac{d^3\phi}{dx^3} - \frac{d\phi}{dx}\cdot\frac{d^2\phi}{dx^2}\right)\Big|_0^6 = 0$, $\left(\frac{d^2\phi}{dx^2}\right)^2 \geq 0$, and $\phi^2 > 0$. It follows that $\lambda \geq 0$.

The SL problem with B.C.'s satisfies:

$$L(\phi) - \lambda\phi = \phi'''' - \lambda\phi = 0, \qquad \phi(0) = 0, \quad \phi''(0) = 0, \quad \phi(6) = 0, \quad \text{and} \quad \phi''(6) = 0.$$

The characteristic equation is $r^4 - \lambda = 0$, so we consider the various cases of real λ.

If $\lambda = 0$, then integrating 4 times gives:

$$\phi(x) = \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4.$$

We have $\phi(0) = c_4 = 0$ and $\phi''(0) = c_2 = 0$. Since $\phi''(6) = 6c_1 = 0$, $c_1 = 0$. Finally, $\phi(6) = 6c_3 = 0$ or $c_3 = 0$, leaving only the trivial solution, so $\lambda = 0$ is not an eigenvalue.

From above it follows that λ must be positive, so assume $\lambda = \alpha^4 > 0$. The characteristic equation satisfies:

$$r^4 - \alpha^4 = (r^2 + \alpha^2)(r^2 - \alpha^2) = (r + i\alpha)(r - i\alpha)(r + \alpha)(r - \alpha) = 0.$$

This gives the general solution:

$$\phi(x) = c_1\cosh(\alpha x) + c_2\sinh(\alpha x) + c_3\cos(\alpha x) + c_4\sin(\alpha x).$$

The BCs $\phi(0) = 0$ and $\phi''(0) = 0$ give:

$$c_1 + c_3 = 0 \quad \text{and} \quad \alpha^2(c_1 - c_3) = 0, \qquad \text{so} \quad c_1 = c_3 = 0.$$

The BCs $\phi(6) = 0$ and $\phi''(6) = 0$ give:

$$c_2\sinh(6\alpha) + c_4\sin(6\alpha) = 0 \quad \text{and} \quad \alpha^2(c_2\sinh(6\alpha) - c_4\sin(6\alpha)) = 0.$$

It follows that $c_2\sinh(6\alpha) = 0$ or $c_2 = 0$ and $c_4\sin(6\alpha) = 0$, which leads to non-trivial solutions if $6\alpha_n = n\pi,\ n = 1, 2, ...$ It follows that the eigenvalues and corresponding eigenfunctions are given by:

$$\lambda_n = \alpha_n^4 = \left(\frac{n\pi}{6}\right)^4 \qquad \text{and} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{6}\right) \ \text{for } n = 1, 2, ...$$

The orthogonality relationship satisfies:

$$\int_0^6 \sin\left(\frac{m\pi x}{6}\right)\sin\left(\frac{n\pi x}{6}\right)dx = 0, \qquad n \neq m.$$

c. The displacement of a uniform thin beam in a medium that resists motion satisfies:

$$\frac{\partial^4 u}{\partial x^4} = -\frac{\partial^2 u}{\partial t^2} - 0.2\frac{\partial u}{\partial t}, \qquad 0 < x < 6, \quad t > 0.$$

with the B.C.'s:

$$u(0,t) = 0, \quad u_{xx}(0,t) = 0, \quad u(6,t) = 0, \quad u_{xx}(6,t) = 0.$$

and I.C.'s, $u(x,0) = 0$, and the initial velocity:

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 0, & x \in (0,1), \\ 2, & x \in (1,2), \\ 0, & x \in (2,6). \end{cases}$$

We use separation of variables with $u(x,t) = \phi(x)h(t)$, so

$$\phi''''h = -\phi(h'' - 0.2h') \qquad \text{or} \qquad \frac{\phi''''}{\phi} = -\frac{h'' - 0.2h'}{h} = \lambda.$$

This gives the SL problem from Part b, $\phi^{(4)} - \lambda\phi = 0$ with BCs $\phi(0) = 0$, $\phi''(0) = 0$, $\phi(6) = 0$, $\phi''(6) = 0$. which has the eigenvalues and eigenfunctions:

$$\lambda_n = \alpha_n^4 = \left(\frac{n\pi}{6}\right)^4 \qquad \text{with} \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{6}\right), \quad n = 1, 2, ...$$

The t-equation becomes:

$$h_n'' + 0.2h_n' + \lambda_n h_n = 0,$$

which has the characteristic equation:

$$r^2 + 0.2r + \lambda_n = 0, \qquad \text{so} \qquad r = -0.1 \pm \sqrt{0.01 - \lambda_n}.$$

Since the smallest eigenvalue is $\lambda_1 = \left(\frac{\pi}{6}\right)^4 \approx 0.0752 > 0.01$, we have $0.01 - \lambda_n < 0$, which implies that r is complex. Thus, if we define $\omega_n = \sqrt{\lambda_n - 0.01}$, then we write the solution:

$$h_n(t) = e^{-0.1t}\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right), \quad n = 1, 2, ...$$

The Superposition Principle gives:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-0.1t}\Big(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\Big) \sin\left(\frac{n\pi x}{6}\right).$$

From the I.C., $u(x,0) = 0$, it follows that $A_n = 0$. The velocity satisfies:

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n e^{-0.1t}\Big(\omega_n \cos(\omega_n t) - 0.1\sin(\omega_n t)\Big) \sin\left(\frac{n\pi x}{6}\right).$$

The other I.C. gives:

$$\frac{\partial u}{\partial t}(x,0) = F(x) = \sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{6}\right).$$

Multiplying by $\phi_n(x)$, integrating from $x = 0$ to 6, and using orthogonality gives:

$$\int_0^6 F(x)\sin\left(\frac{n\pi x}{6}\right)dx = \int_0^6 B_n\omega_n \sin^2\left(\frac{n\pi x}{6}\right)dx = 3B_n\omega_n.$$

It follows that the Fourier coefficient is:

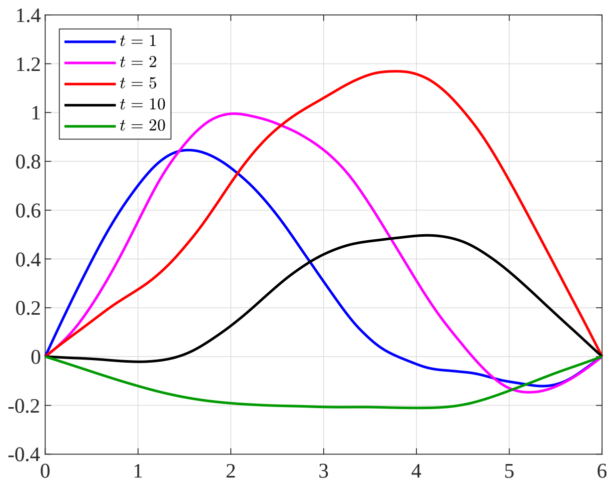
$$\begin{aligned} B_n &= \frac{1}{3\omega_n}\int_0^6 F(x)\sin\left(\frac{n\pi x}{6}\right)dx = \frac{2}{3\omega_n}\int_1^2 \sin\left(\frac{n\pi x}{6}\right)dx, \\ &= -\frac{4}{n\pi\omega_n}\cos\left(\frac{n\pi x}{6}\right)\Big|_1^2 = \frac{4\Big(\cos\left(\frac{n\pi}{6}\right) - \cos\left(\frac{n\pi}{3}\right)\Big)}{n\pi\omega_n}. \end{aligned}$$

Thus, the solution to this problem is:

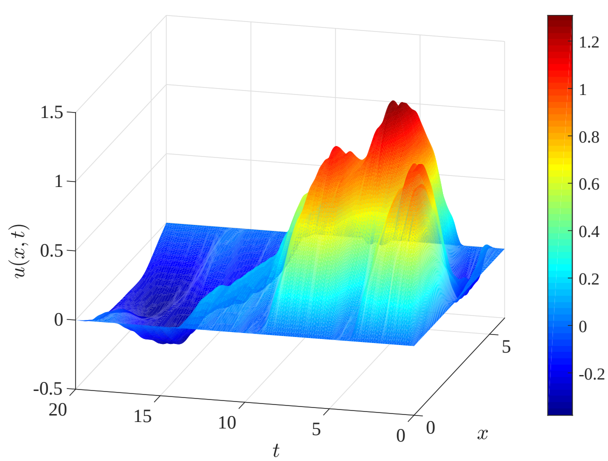
$$u(x,t) = \sum_{n=1}^{\infty} \frac{4\Big(\cos\left(\frac{n\pi}{6}\right) - \cos\left(\frac{n\pi}{3}\right)\Big)}{n\pi\omega_n} e^{-0.1t}\sin(\omega_n t)\sin\left(\frac{n\pi x}{6}\right),$$

where $\omega_n = \sqrt{\left(\frac{n\pi}{6}\right)^4 - 0.01}$.

d. With 50 terms in the series solution of $u(x,t)$, the figure below shows the displacement of the beam at times $t = 0$, 1, 2, 5, 10, and 20.



Below is a surface plot for $u(x,t)$ with $x \in [0,6]$ and $t \in [0,50]$.



```
%format compact;
L = 6;
NptsX=151;          % number of x pts
NptsT=151;          % number of t pts
Nf=200;              % number of Fourier terms
x=linspace(0,L,NptsX);
t=linspace(0,20,NptsT);
[X,T]=meshgrid(x,t);
k = 0.2;

fs=8;
figure(1)
clf

b=zeros(1,Nf);
U=zeros(NptsT,NptsX);
for n=1:Nf
    w(n) = sqrt((n*pi/L)^4-0.01);
    b(n) = (4*(cos(n*pi/L)-cos(2*n*pi/L)))/(n*pi*w(n));
    Un = b(n)*exp(-(k/2)*T).*sin(w(n)*T).*sin(n*pi*X/L);
    U = U + Un;
end
set(gca,'FontSize',[fs]);
surf(X,T,U);
shading interp
colormap(jet)
fontlabs = 'Times New Roman';
xlabel('$x$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
ylabel('$t$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
%axis tight
colorbar
view([-75 20])
set(gca,'FontSize',12);          % Axis tick font size
print -depsc beam_plots20.eps

figure(2)

xx=linspace(0,6,200);
V1=zeros(1,200);
for n=1:Nf
    Vn = b(n)*exp(-(k/2)*1)*sin(w(n)*1).*sin(n*pi*xx/L);
    V1 = V1 + Vn;
end
V2=zeros(1,200);
for n=1:Nf
    Vn = b(n)*exp(-(k/2)*2)*sin(w(n)*2).*sin(n*pi*xx/L);
    V2 = V2 + Vn;
end
V5=zeros(1,200);
for n=1:Nf
    Vn = b(n)*exp(-(k/2)*5)*sin(w(n)*5).*sin(n*pi*xx/L);
    V5 = V5 + Vn;
end
V10=zeros(1,200);
for n=1:Nf
    Vn = b(n)*exp(-(k/2)*10)*sin(w(n)*10).*sin(n*pi*xx/L);
    V10 = V10 + Vn;
end
```

```
V20=zeros(1,200);
for n=1:Nf
    Vn = b(n)*exp(-(k/2)*20)*sin(w(n)*20).*sin(n*pi*xx/L);
    V20 = V20 + Vn;
end
plot(xx,V1,'b-','LineWidth',1.5);
hold on
plot(xx,V2,'m-','LineWidth',1.5);
plot(xx,V5,'r-','LineWidth',1.5);
plot(xx,V10,'k-','LineWidth',1.5);
plot(xx,V20,'-','color',[0,0.6,0],'LineWidth',1.5);
grid;
h = legend('$t = 1$','$t = 2$','$t = 5$','$t = 10$',...
    '$t = 20$','Location','northwest');
set(h,'Interpreter','latex')
h.FontSize = 10;
xlim([0,6]);
ylim([-0.4,1.4]);
set(gca,'FontSize',12);          % Axis tick font size

print -depsc beam_plotbs20.eps
```