Computer Problem

1. a. (5pts) Consider a one-dimensional rod that is insulated along its edges. Assume that it has a length of 10 cm. The rod is initially placed so that one end is 0°C and the other end is 100°C. It is allowed to come to a steady-state temperature distribution. Find this temperature distribution, $u_e(x)$.

For steady state temperature distribution, we have $u(x,t) = u_e(x)$. Thus, $u_e'' = 0$, which integrates twice to give:

$$u_e(x) = c_1 x + c_2.$$

The b.c. $u_e(0) = 0$ implies that $c_2 = 0$, while the b.c. $u_e(10) = 100 = c_1 \cdot 10$ or $c_1 = 10$. It follows that

$$u_e(x) = 10x.$$

b. (25pts) At time $t = 0$, the one-dimensional rod from Part a is insulated on both ends. This implies that the rod satisfies the PDE:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad t > 0, \quad 0 < x < 10, \\ \text{Boundary Conditions :} &\quad \frac{\partial u(0,t)}{\partial x} = 0, \qquad \frac{\partial u(10,t)}{\partial x} = 0, \quad t > 0, \\ \text{Initial Conditions :} &\quad u(x,0) = 10x, \quad 0 < x < 10. \end{aligned}$$

Create a graphic simulation showing the 3D plot of temperature as a function of t and x, using 5 and 50 terms (Fourier coefficients) to approximate the solution with $t \in [0, 20]$.

Let $u(x,t) = \phi(x)G(t)$, then separation of variables gives:

$$G'\phi = \phi'' G \qquad \text{or} \qquad \frac{G'}{G} = \frac{\phi''}{\phi} = -\lambda.$$

The time varying solution of $G' = -\lambda G$ is $G(t) = ce^{-\lambda t}$. The Sturm-Liouville problem satisfies

$$\phi'' + \lambda\phi = 0, \qquad \text{with} \qquad \phi'(0) = \phi'(10) = 0.$$

If $\lambda = -\alpha^2 < 0$, then $\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$. The b.c. $\phi'(0) = 0$ implies $c_2 = 0$, then $\phi'(10) = 0$ yields $c_1 = 0$, giving only the trivial solution.

If $\lambda_0 = 0$, then $\phi(x) = c_1 x + c_2$. Both b.c.s give $c_1 = 0$, so c_2 is arbitrary. It follows that $\lambda_0 = 0$ is an eigenvalue with corresponding eigenfunction, $\phi_0(x) = 1$.

If $\lambda = \alpha^2 > 0$, then $\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. The b.c. $\phi'(0) = 0$ implies $c_2 = 0$. For nontrivial solutions, the b.c. $\phi'(10) = 0$ gives $\alpha = \frac{n\pi}{10}$. It follows that we have eigenvalues and eigenfunctions:

$$\lambda_n = \frac{n^2\pi^2}{100} \qquad \text{with} \qquad \phi_n(x) = \cos\left(\frac{n\pi x}{10}\right).$$

By the superposition principle, we obtain the solution:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{100}t} \cos\left(\frac{n\pi x}{10}\right).$$

The initial conditions give:

$$u(x,0) = 10x = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{10}\right). \tag{1}$$

We use the orthogonality of the eigenfunctions to obtain the Fourier coefficients. If we multiply (1) by $\phi_0(x)$ and integrate from 0 to 10, we have:

$$\begin{aligned}
\int_0^{10} 10x\,dx &= \int_0^{10} A_0\,dx, \\
500 &= 10A_0, \\
A_0 &= 50.
\end{aligned}$$

Similarly, we multiply (1) by $\phi_n(x)$ and integrate from 0 to 10, then orthogonality gives:

$$\begin{aligned}
\int_0^{10} 10x\cos\left(\frac{n\pi x}{10}\right)dx &= \int_0^{10} A_n \cos^2\left(\frac{n\pi x}{10}\right)dx, \\
5A_n &= 10\left[x\cdot\frac{10}{n\pi}\sin\left(\frac{n\pi x}{10}\right)\Big|_0^{10} - \int_0^{10}\frac{10}{n\pi}\sin\left(\frac{n\pi x}{10}\right)dx\right] \\
A_n &= 2\left[\frac{100}{n^2\pi^2}\cos\left(\frac{n\pi x}{10}\right)\Big|_0^{10}\right] \\
A_n &= \left(\frac{200}{n^2\pi^2}\right)\left[\cos(n\pi) - 1\right] \\
A_n &= \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right].
\end{aligned}$$

Thus,

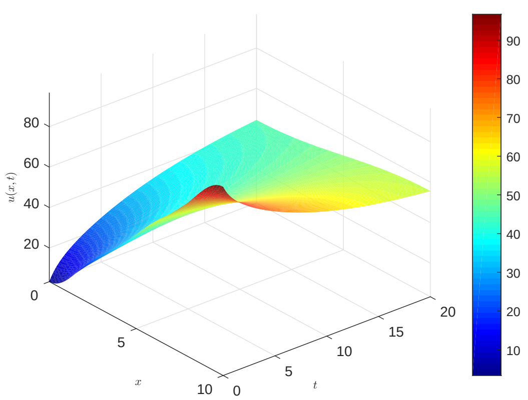
$$u(x,t) = 50 + \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} e^{-\frac{n^2\pi^2}{100}t}\cos\left(\frac{n\pi x}{10}\right).$$

Below is the MatLab code for drawing the surface with 5 and 50 Fourier coefficients:

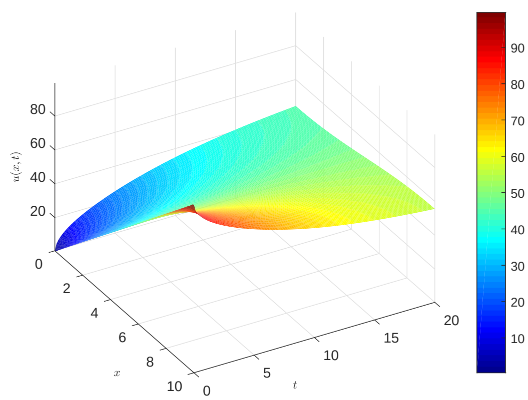
```
format compact;
W = 10;                 % width of plate
tfin = 20;              % final time
alpha = 1;              % heat coef of the medium

NptsX=151;              % number of x pts
NptsT=151;              % number of t pts
Nf=5;                  % number of Fourier terms
x=linspace(0,W,NptsX);
t=linspace(0,tfin,NptsT);
[X,T]=meshgrid(x,t);

```

```
fs=8;
figure(1)
clf

b=zeros(1,Nf);
U=zeros(NptsT,NptsX)+50;

for n=1:Nf
    b(n)=(200/(n^2*pi^2))*((-1)^n - 1);  % Fourier coefficients
    Un=b(n)*exp(-(n*pi*alpha/W)^2*T).*cos(n*pi*X/W);  % Temperature(n)
    U=U+Un;
end
set(gca,'FontSize',[fs]);
surf(X,T,U);
shading interp
colormap(jet)
fontlabs = 'Times New Roman';
xlabel('$x$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
ylabel('$t$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
zlabel('$u(x,t)$','Fontsize',fs,'FontName',fontlabs,'interpreter','latex');
axis tight
colorbar
view([50 33])
print -depsc heat5.eps
```

This program is used to produce the following heat surfaces showing the evolution of the heat equation with 5 and 50 Fourier coefficients.



Heat Map with 5 Fourier terms



Heat Map with 50 Fourier terms