1. (1pts) a. Consider the model below:

$$\begin{aligned} \dot{x}_1 &= \tfrac{13}{10}x_2 - \tfrac{7}{5}x_1 - \tfrac{6}{5}, \\ \dot{x}_2 &= \tfrac{1}{10}x_1 - \tfrac{1}{5}x_2 + \tfrac{9}{5}. \end{aligned} \tag{1}$$

With $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, A be a 2×2 matrix, and $\mathbf{b}$ be a 2×1, it is apparent that this can be written into a matrix system of the form: $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$, as follows:

$$\dot{\mathbf{x}} = \begin{pmatrix} -\frac{7}{5} & \frac{13}{10} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\frac{6}{5} \\ \frac{9}{5} \end{pmatrix}.$$

b. (2pts) To find the equilibria for (1), we solve the linear system, $\dot{\mathbf{x}} = \mathbf{0}$ or

$$\begin{pmatrix} -\frac{7}{5} & \frac{13}{10} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}\mathbf{x_e} = \begin{pmatrix} \frac{6}{5} \\ -\frac{9}{5} \end{pmatrix}.$$

We use row reduction as follows:

$$\left[\begin{array}{ccc} -\frac{7}{5} & \frac{13}{10} & : & \frac{6}{5} \\ \frac{1}{10} & -\frac{1}{5} & : & -\frac{9}{5} \end{array}\right] \quad \begin{array}{c} -\frac{5}{7}R_1 \\ \longrightarrow \\ 10R_2 \end{array} \quad \left[\begin{array}{cccc} 1 & -\frac{13}{14} & : & -\frac{6}{7} \\ 1 & -2 & : & -18 \end{array}\right]$$

$$\begin{array}{c} R_2 - R_1 \\ \longrightarrow \end{array} \quad \left[\begin{array}{cccc} 1 & -\frac{13}{14} & : & -\frac{6}{7} \\ 0 & -\frac{15}{14} & : & -\frac{120}{7} \end{array}\right] \quad \begin{array}{c} -\frac{14}{15}R_2 \\ \longrightarrow \end{array} \quad \left[\begin{array}{cccc} 1 & -\frac{13}{14} & : & -\frac{6}{7} \\ 0 & 1 & : & 16 \end{array}\right]$$

$$\begin{array}{c} R_1 + \frac{13}{14}R_2 \\ \longrightarrow \end{array} \quad \left[\begin{array}{cccc} 1 & 0 & : & 14 \\ 0 & 1 & : & 16 \end{array}\right] \quad \text{or} \quad x_e = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$$

c. (1pts) Let $\mathbf{y} = \mathbf{x} - \mathbf{x}_e$, so $\dot{\mathbf{y}} = \dot{\mathbf{x}}$. We substitute into (1) giving:

$$\dot{\mathbf{y}} = \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b} = A(\mathbf{y} + \mathbf{x}_e) + \mathbf{b} = A\mathbf{y} + (A\mathbf{x}_e + \mathbf{b}) = A\mathbf{y},$$

creating the homogeneous linear system,

$$\dot{\mathbf{y}} = A\mathbf{y}. \tag{2}$$

The equilibrium for this system satisfies, $A\mathbf{y}_e = \mathbf{0}$. However, A is nonsingular, so it follows that

$$y_e = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

d. (4pts) We attempt a solution of the form, $\mathbf{y}(t) = \xi e^{\lambda t}$ in (2). The result is:

$$\lambda \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} -\frac{7}{5} & \frac{13}{10} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t} \quad \text{or} \quad \begin{pmatrix} -\frac{7}{5} - \lambda & \frac{13}{10} \\ \frac{1}{10} & -\frac{1}{5} - \lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This eigenvalue problem gives the characteristic equation:

$$\begin{vmatrix} -\frac{7}{5}-\lambda & \frac{13}{10} \\ \frac{1}{10} & -\frac{1}{5}-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad \lambda^2 + \tfrac{8}{5}\lambda + \tfrac{15}{100} = \left(\lambda + \tfrac{1}{10}\right)\left(\lambda + \tfrac{15}{10}\right) = 0.$$

It follows that the eigenvalues are:

$$\lambda_1 = -\tfrac{1}{10} \qquad \text{and} \qquad \lambda_2 = -\tfrac{3}{2}.$$

For $\lambda_1 = -\frac{1}{10}$, the associated eigenvector satisfies:

$$\begin{pmatrix} -\frac{13}{10} & \frac{13}{10} \\ \frac{1}{10} & -\frac{1}{10} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0, \qquad \text{so} \qquad \xi_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -\frac{3}{2}$, the associated eigenvector satisfies:

$$\begin{pmatrix} \frac{1}{10} & \frac{13}{10} \\ \frac{1}{10} & \frac{13}{10} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0, \qquad \text{so} \qquad \xi_2 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -13 \\ 1 \end{pmatrix}.$$

e. (4pts) From the eigenvalue problem, we obtain the general solution to (2):

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/10} + c_2 \begin{pmatrix} -13 \\ 1 \end{pmatrix} e^{-3t/2}.$$

The solution $\mathbf{x}(t) = \mathbf{y}(t) + \mathbf{x}_e$ or

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/10} + c_2 \begin{pmatrix} -13 \\ 1 \end{pmatrix} e^{-3t/2} + \begin{pmatrix} 14 \\ 16 \end{pmatrix}.$$

We apply the IC to find the unique solution to the IVP for (1), so

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -13 \\ 1 \end{pmatrix} + \begin{pmatrix} 14 \\ 16 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \end{pmatrix}.$$

Thus, we solve the linear problem:

$$\begin{pmatrix} 1 & -13 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -8 \\ 8 \end{pmatrix}.$$

We use row reduced echelon form to solve this as follows:

$$\left[\begin{array}{cc:c} 1 & -13 & -8 \\ 1 & 1 & 8 \end{array}\right] \quad \overset{R_2 - R_1}{\longrightarrow} \quad \left[\begin{array}{cc:c} 1 & -13 & -8 \\ 0 & 14 & 16 \end{array}\right]$$

$$\overset{\frac{1}{14}R_2}{\longrightarrow} \quad \left[\begin{array}{cc:c} 1 & -13 & -8 \\ 0 & 1 & \frac{8}{7} \end{array}\right] \quad \overset{-13R_2}{\longrightarrow} \quad \left[\begin{array}{cc:c} 1 & 0 & \frac{48}{7} \\ 0 & 1 & \frac{8}{7} \end{array}\right].$$

Thus, the unique solution to (2) satisfies:

$$\mathbf{x}(t) = \tfrac{48}{7} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/10} + \tfrac{8}{7} \begin{pmatrix} -13 \\ 1 \end{pmatrix} e^{-3t/2} + \begin{pmatrix} 14 \\ 16 \end{pmatrix}.$$

f. (3pts) This unique solution gives:

$$x_1(t) = \tfrac{48}{7}e^{-t/10} - \tfrac{104}{7}e^{-3t/2} + 14,$$

which has a horizontal asymptote at $x_1(t) = 14$, and

$$x_2(t) = \tfrac{48}{7}e^{-t/10} + \tfrac{8}{7}e^{-3t/2} + 16,$$

which has a horizontal asymptote at $x_2(t) = 16$.

With MatLab we graph the solutions, $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, for $t \in [0, 20]$, showing the horizontal asymptotes for these solutions.