1. (5pts) The 3^{rd} order linear homogeneous ODE given by:

$$t^2 y''' - t y'' + 2y' = 0,$$

uses similar techniques for solving this *Cauchy-Euler problem* as done in class. We attempt solutions of the form $y(t) = t^r$ and substitute into the original equation:

$$t^2 r(r-1)(r-2)t^{r-2} - tr(r-1)t^{r-1} + 2rt^{r-1} = t^{r-1}r\big((r-1)(r-2)-(r-1)+2\big) = t^{r-1}r(r^2-4r+5) = 0.$$

This gives the *auxiliary equation*, $r(r^2 - 4r + 5) = 0$, which has roots, $r = 0, 2 \pm i$. It follows that the real three solutions are:

$$y_1(t) = 1, \qquad y_2(t) = t^2\cos(\ln|t|), \qquad y_3(t) = t^2\sin(\ln|t|).$$

The easiest way to determine that these solutions are linearly independent is to directly use the definition of linearly independent at three different times. That is consider

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t),$$

at three times, then if $y(t_1) = 0$, $y(t_2) = 0$, and $y(t_3) = 0$ imply that $c_1 = c_2 = c_3 = 0$. Below we show the matrix from our solutions and $t = 1$, 2, and 3 is nonsingular, which is equivalent to linear independence.

$$Y = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4\cos(\ln(2)) & 4\sin(\ln(2)) \\ 1 & 9\cos(\ln(3)) & 9\sin(\ln(3)) \end{pmatrix}, \qquad \det|Y| \approx 8.74,$$

so Y is invertible. There is a general theory that a generalized Wronskian being nonzero shows this linear independence, but that requires more work.

2. (5pts) The following ODE:

$$y'' + 16\,y = 32\csc^2(4t),$$

is solved using the Variation of parameters method. From the homogeneous part, the characteristic equation for this problem satisfies $\lambda^2 + 16 = 0$, which gives $\lambda = \pm 4i$. It follows that the solution to the homogeneous problem is:

$$y_h(t) = c_1\cos(4t) + c_2\sin(4t).$$

Computing the Wronskian, we see:

$$W\left[\cos(4t), \sin(4t)\right] = \begin{vmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{vmatrix} = 4.$$

Variation of parameters formula gives us

$$\begin{aligned} y_p(t) &= -\cos(4t)\int^t \frac{32\csc^2(4s)\sin(4s)}{4}ds + \sin(4t)\int^t \frac{32\csc^2(4s)\cos(4s)}{4}ds, \\ &= -8\cos(4t)\int^t \csc(4s)ds + 8\sin(4s)\int^t \cot(4s)\csc(4s)ds, \\ &= 2\cos(4t)\ln|\csc(4t)+\cot(4t)| - 2\sin(4t)\csc(4t), \\ &= 2\cos(4t)\ln|\csc(4t)+\cot(4t)| - 2. \end{aligned}$$

It follows that the general solution is given by:

$$y(t) = c_1 \cos(4t) + c_2 \sin(4t) + 2\cos(4t)\ln|\csc(4t) + \cot(4t)| - 2.$$

3. (6pts) a. Consider the linear homogeneous ODE given by:

$$ty'' - y' + 4t^3 y = 0.$$

We are to show that $y_1(t) = \cos(t^2)$ and $y_2(t) = \sin(t^2)$ are solutions to this ODE. Computing the derivatives of $y_1(t)$ gives:

$$y_1'(t) = -2t\sin(t^2), \qquad y_1''(t) = -2\sin(t^2) - 4t^2\cos(t^2).$$

Inserting these into our original ODE gives:

$$-2t\sin(t^2) - 4t^3\cos(t^2) + 2t\sin(t^2) + 4t^3\cos(t^2) = 0.$$

Computing the derivatives of $y_2(t)$ gives:

$$y_2'(t) = 2t\cos(t^2), \qquad y_2''(t) = 2\cos(t^2) - 4t^2\sin(t^2).$$

Inserting these into our original ODE gives:

$$2t\cos(t^2) - 4t^3\sin(t^2) - 2t\cos(t^2) + 4t^3\sin(t^2) = 0.$$

Thus $y_1(t)$ and $y_2(t)$ are solutions.

Next find the *Wronskian* of these solutions, $W[y_1, y_2](t)$ and show it is nonzero to prove that these solutions form a *fundamental set of solutions* to this ODE. The Wronskian satisfies.

$$W_{[y_1,y_2]} = \begin{vmatrix} \cos(t^2) & \sin(t^2) \\ -2t\sin(t^2) & 2t\cos(t^2) \end{vmatrix} = 2t(\cos^2(t^2) + \sin^2(t^2)) = 2t,$$

which is nonzero provided $t \neq 0$. Thus, if $t \neq 0$, then y_1 and y_2 form a fundamental set of solutions, and the general solution has the form:

$$y(t) = c_1\cos(t^2) + c_2\sin(t^2).$$

b. Consider the linear nonhomogeneous ODE given by:

$$ty'' - y' + 4t^3 y = 8t^3 \qquad \text{or} \qquad y'' - \frac{1}{t}y' + 4t^2 y = 8t^2.$$

Part a gives the solution to the homogeneous problem, so we use the *Variation of Parameters* method to solve this problem (Slide 24). The particular solution satisfies:

$$\begin{aligned} y_p &= -\cos(t^2)\int^t \frac{\sin(s^2)8s^2}{2s}ds + \sin(t^2)\int^t \frac{\cos(s^2)8s^2}{2s}ds \\ &= -4\cos(t^2)\int^t \sin(s^2)s\,ds + 4\sin(t^2)\int^t \cos(s^2)s\,ds \\ &= -4\cos(t^2)\left(\frac{-\cos(t^2)}{2}\right) + 4\sin(t^2)\left(\frac{\sin(t^2)}{2}\right) \;=\; 2\left(\cos^2(t^2) + \sin^2(t^2)\right) \;=\; 2. \end{aligned}$$

It follows that the general solution is given by:

$$y(t) = c_1\cos(t^2) + c_2\sin(t^2) + 2.$$