This is a collection of some of the HW problems worked as examples.

1. Consider
$$\frac{dy}{dx} = xy + 4x + 2y + 8 = (x+2)(y+4),$$
which is both linear and separable. As a linear problem, we write
$$\frac{dy}{dx} - (x+2)y = 4(x+2), \quad \text{with integrating factor} \quad \mu(x) = exp\left(-\int(x+2)dx\right) = e^{-\left(\frac{x^2}{2}+2x\right)}.$$
Thus,
$$\frac{d}{dx}\left(e^{-\left(\frac{x^2}{2}+2x\right)}y\right) = 4(x+2)e^{-\left(\frac{x^2}{2}+2x\right)}, \qquad \text{or} \qquad e^{-\left(\frac{x^2}{2}+2x\right)}y(x) = -4e^{-\left(\frac{x^2}{2}+2x\right)} + k.$$
It follows that
$$y(x) = -4 + ke^{\left(\frac{x^2}{2}+2x\right)}.$$
With separation of variables, we have
$$\int \frac{dy}{y+4} = \int (x+2)dx \qquad \text{or} \qquad \ln|y+4| = \frac{x^2}{2} + 2x + C.$$
Solving we obtain the same solution as above with $k = e^C$.

2. Consider the following IVP:
$$t\frac{dy}{dt} = 60t - 5y - 45, \qquad y(1) = 10.$$
Dividing by $t$ and rearranging gives the standard form:
$$\frac{dy}{dt} + \frac{5}{t}y = 60 - \frac{45}{t}, \quad \text{so} \quad \mu(t) = e^{\int \frac{5}{t}dt} = t^5.$$
It follows that
$$\frac{d}{dt}\left(t^5 y\right) = 60t^5 - 45t^4, \quad \text{or} \quad t^5 y(t) = \int \left(60t^5 - 45t^4\right)dt = 10t^6 - 9t^5 + C.$$
The IC gives $10 = 10 + -9 + C$ or $C = 9$, so the unique solution is
$$y(t) = 10t - 9 + \frac{9}{t^5}.$$

3. Consider the following IVP:
$$\frac{dy}{dt} = 12t + 6.3e^{0.9t}, \qquad y(0) = 3.$$

The right hand side of this ODE is only a function of $t$, so we just integrate to obtain the solution. Thus, we have:

$$y(t) = \int \left(12t + 6.3e^{0.9t}\right)dt = 6t^2 + 7e^{0.9t} + C.$$

The IC gives $3 = 7 + C$ or $C = -4$, so the unique solution is

$$y(t) = 6t^2 + 7e^{0.9t} - 4.$$

4. Consider the following IVP:

$$t\frac{dy}{dt} - 3y = 36t^4\cos(4t), \qquad y(1) = 7.$$

Dividing by $t$ gives the standard form:

$$\frac{dy}{dt} - \frac{3}{t}y = 36t^3\cos(4t), \quad \text{so} \quad \mu(t) = e^{-\int \frac{3}{t}dt} = t^{-3}.$$

It follows that

$$\frac{d}{dt}\left(t^{-3}y\right) = 36\cos(4t), \quad \text{or} \quad t^{-3}y(t) = 36\int \cos(4t)dt = 9\sin(4t) + C.$$

The IC gives $7 = 9\sin(4) + C$ or $C = 7 - 9\sin(4)$, so the unique solution is

$$y(t) = t^3\left(9\sin(4t) + 7 - 9\sin(4)\right).$$

5. The equation $\frac{dy}{dt} = 3t^2 + 12$ is an integrable differential equation, so

$$y(t) = \int (3t^2 + 12)dt = t^3 + 12t + C.$$

With the initial condition, $y(0) = 8 = C$. It follows that

$$y(t) = t^3 + 12t + 8.$$

6. Consider the following IVP:

$$\frac{dy}{dt} - 4\tan(4t)y = 40\sin^4(4t), \qquad y(0) = 14.$$

The integrating factor satisfies:

$$\mu(t) = e^{-4\int \tan(4t)dt} = e^{-\int \frac{4\sin(4t)}{\cos(4t)}dt} = e^{\ln(\cos(4t))} = \cos(4t).$$

It follows that

$$\frac{d}{dt}\Big(\cos(4t)y\Big) = 40\sin^4(4t)\cos(4t), \quad \text{or} \quad \cos(4t)y(t) = 40\int \sin^4(4t)\cos(4t)dt.$$

Use $u = \sin(4t)$ and $du = 4\cos(4t)dt$, then

$$\cos(4t)y(t) = 10\int u^4 du = 2\sin^5(4t) + C, \quad \text{or} \quad y(t) = \sec(4t)\Big(2\sin^5(4t) + C\Big).$$

The IC gives $14 = C$, so the unique solution is

$$y(t) = \sec(4t)\Big(2\sin^5(4t) + 14\Big).$$

7. a. According to the von Bertalanffy equation, the fish growth satisfies $\frac{dL}{dt} = k(34 - L(t)) = -k(L(t) - 34), \quad L(0) = 2$. To solve this we make the substitution, $z(t) = L(t) - 34$, which has $z(0) = -32$ and $\frac{dz}{dt} = \frac{dL}{dt}$. The modified differential equation becomes $z' = -kz$, which has the solution $z(t) = -32e^{-kt} = L(t) - 34$. Thus, the length of the fish satisfies:

$$L(t) = 34 - 32e^{-kt}.$$

b. If $L(4) = 10$, then $34 - 32e^{-4k} = 10 \quad$ or $\quad e^{4k} = \frac{32}{24} = \frac{4}{3}$. It follows that $k = \frac{1}{4}\ln\left(\frac{4}{3}\right) \approx 0.0719205$. The length of the fish satisfies:

$$L(t) = 34 - 32e^{-0.0719205t}.$$

c. When $t = 10$, $L(t) = 34 - 32e^{-0.0719205\cdot 10} \approx 18.4115$ cm. Since the exponential decays to zero as $t \to \infty$, $L(t) \to 34$ cm.

8. a. Let $a(t)$ be the amount of pollutant, and the concentration $c(t)$ is the concentration of pollutant (in ppb). The change in amount = the amount entering - the amount leaving. The change in amount, $a'(t)$, has units (mass/day). The amount entering is $f_1Q_1 + f_2Q_2 = 4000\cdot 18 + 2500\cdot 4 = 82,000$ ppb·m$^3$/day (mass/day), while the amount leaving is $(f_1 + f_2)c(t) = 6500c(t)$ ppb·m$^3$/day (mass/day). Thus, the differential equation for the change in amount is

$$\frac{da(t)}{dt} = 82,000 - 6500c(t).$$

The relation between the amount and concentration is $c(t) = \frac{a(t)}{V} = \frac{a(t)}{3000000}$ and $c'(t) = \frac{a'(t)}{3000000}$, so the concentration differential equation:

$$\frac{dc(t)}{dt} = \frac{82,000 - 6500c(t)}{3,000,000} - \frac{13}{6000}\left(c - \frac{164}{13}\right) \approx -0.0021667(c - 12.615).$$

With the initial condition $c(0) = 0$, we make the substitution $z(t) = c(t) - 12.615$, so $z(0) = -12.615$ and the differential equation is

$$\frac{dz}{dt} = -0.0021667z, \qquad z(0) = -12.615,$$

which gives $z(t) = -12.615e^{-0.0021667t} = c(t) - 12.615$. Thus,

$$c(t) = 12.615\left(1 - e^{-0.0021667t}\right).$$

b. We solve $c(t) = 12.615\left(1 - e^{-0.0021667t}\right) = 4$, so $e^{0.0021667t} = \frac{12.615}{8.615} = 1.4643$. Thus, $t = \frac{\ln(1.4643)}{0.0021667} \approx 176.01$ days. Hence, the concentration reaches 4 ppb at $t \approx 176.01$ days. The limiting concentration is

$$\lim_{t\to\infty} c(t) = 12.615 \text{ ppb}.$$

This easily follows because the exponential tends to zero for large $t$.

9. The electric circuit with $R = 10\Omega$, $C = 0.1\text{F}$, and $E(t) = 40\text{V}$ is a linear differential equation:

$$10\frac{dQ}{dt} + \frac{1}{0.1}Q = 40 \qquad \text{or} \qquad \frac{dQ}{dt} + Q = 4,$$

which has an integrating factor $\mu(t) = e^t$. Thus,

$$\frac{d}{dt}\left(e^t Q\right) = 4e^t, \quad \text{so} \quad e^t Q(t) = 4e^t + C.$$

With the initial condition, $Q(0) = 0$, the solution becomes

$$Q(t) = 4 - 4e^{-t}.$$

10. Programs are provided to solve this problem along with solutions expected in the written work.

a. The least squares best fit to the data is found with MatLab. The sum of square errors is computed with:

```
function J = sum_vonB(p,tdata,ldata)
% von Bertalanffy eqn
model = p(1)*(1 - exp(-p(2)*tdata));  % Model eqn with parameters
error = model - ldata;      % Error between model and data
J = error*error';           % Sum of square errors
end
```

The least sum of square errors uses the command line:
`[p1,J,flag] = fminsearch(@sum_vonB,[1.9,0.2],[],tdfish,ldfish)`
where `tdfish` and `ldfish` are the age and length data for the fish.

Below is a MatLab program that finds all values needed for this problem and creates all the graphs.

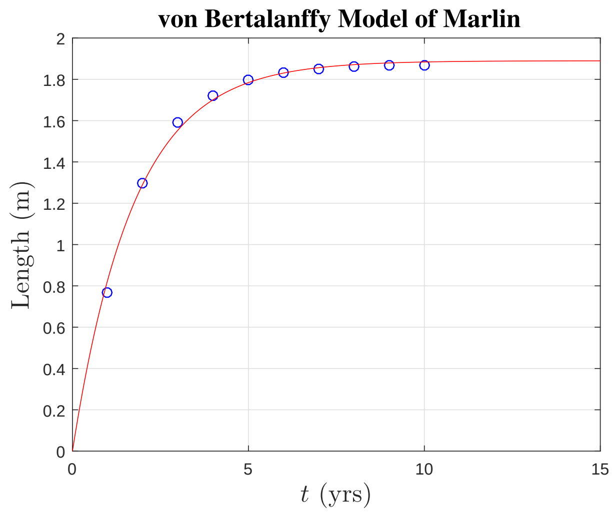
```
% von Bertalanffy problem

clear;clc;

td = [1 2 3 4 5 6 7 8 9 10];     % Age data
ld = [0.77 1.3 1.59 1.72 1.8 1.83 1.85 1.86 1.87 1.87];  % Length data
[p1,J] = fminsearch(@sum_vonB,[1.88,0.5],[],td,ld) % Compute least SSE
tt = linspace(0,15,200);
ll = p1(1)*(1 - exp(-p1(2)*tt));  % Fish model
plot(td,ld,'bo');                 % Plot data
hold on
plot(tt,ll,'r-');grid;            % Plot model
title('von Bertalanffy Model of Marlin','FontSize',16,'FontName','Times New ...
    Roman');
xlabel('$t$ (yrs)','FontSize',16,'interpreter','latex');
ylabel('Length (m)','FontSize',16,'interpreter','latex');
```

```
hold off

print -depsc marlin_len_gr.eps

figure(102)

Ld = [1 1.11 1.16 1.21 1.28 1.38 1.47 1.53 1.68 1.77];
Wd = [7 12 15 19 21 31 32 34 52 65];
lnLd = log(Ld);
lnWd = log(Wd);
coef = polyfit(lnLd,lnWd,1) %Linear fit to log of data
a = coef(1);
k = exp(coef(2));
Ll = linspace(0,2,200);
Ww = k*Ll.^a;                % Allometric model
plot(Ld,Wd,'ro');            % Plot data
hold on
plot(Ll,Ww,'m-');grid;       % Plot model
xlim([0,2]);
ylim([0,100]);
title('Allometric Model of Marlin','FontSize',16,'FontName','Times New Roman');
xlabel('Length (m)','FontSize',16,'interpreter','latex');
ylabel('Weight (kg)','FontSize',16,'interpreter','latex');
hold off

print -depsc marlin_allo_gr.eps

figure(103)

Wt = 82.7983*(1-exp(-.5764*tt)).^3.6166;  % Composite function
plot(tt,Wt,'b-');    % Plot composite function
grid;
xlim([0,15]);
ylim([0,100]);
title('Weight of Marlin','FontSize',16,'FontName','Times New Roman');
xlabel('Age (yr)','FontSize',16,'interpreter','latex');
ylabel('Weight (kg)','FontSize',16,'interpreter','latex');
hold off

print -depsc marlin_wt_gr.eps

figure(104)

% Insert function for derivative/growth of fish
Wpt = 172.6020184*(1-exp(-.5764*tt)).^2.6166.*exp(-.5764*tt);
plot(tt,Wpt,'-','color',[0,0.6,0]);  %Plot growth
grid;
xlim([0,15]);
ylim([0,25]);
title('Growth of Marlin','FontSize',16,'FontName','Times New Roman');
xlabel('Age (yr)','FontSize',16,'interpreter','latex');
ylabel('Growth Rate (kg/yr)','FontSize',16,'interpreter','latex');
hold off

print -depsc marlin_wpt_gr.eps
```

The $L$-intercept is $L(0) = 0$, and the horizontal asymptote is $L_\infty$.

b. The graph of the best fitting model using the von Bertalanffy model for the Striped Marlin with the data set is seen below, fitting the model

$$L(t) = 1.8898\left(1 - e^{-0.5764t}\right).$$

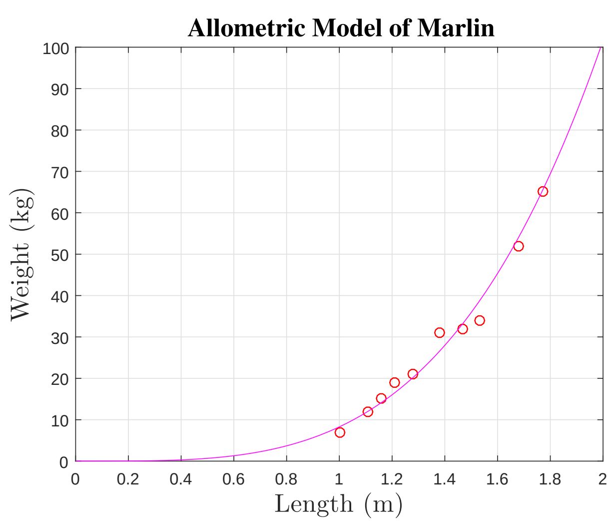



The graph shows that the rate of growth of the fish is very fast in the early years and then slows down, approaching an asymptote, as the fish ages. The model quite clearly matches the data very well. The graph shows that the maximum length the fish is the asymptotic limit, which is about 1.89 m.

c. The Matlab script in Part a shows how to obtain the allometric model by finding the linear least squares best fit to the logarithms of the data.

d. The graph of the best fitting allometric model for the Striped Marlin with the data set is seen below, using the model:

$$W(L) = 8.2861L^{3.6166}.$$




The graph and the model show that the allometric model roughly follows a cubic relationship between the length and the weight, which is expected based on dimensional analysis (between length and volume). The model quite clearly matches the data very well with a little more variation than seen in the previous graph. With the asymptotic limit of the length of the Striped Marlin from Part a, this model would indicate that the Striped Marlin has a limit in weight of approximately 82.8 kg.

e. The composite function simply combines the von Bertalanffy model with the allometric model, yielding:

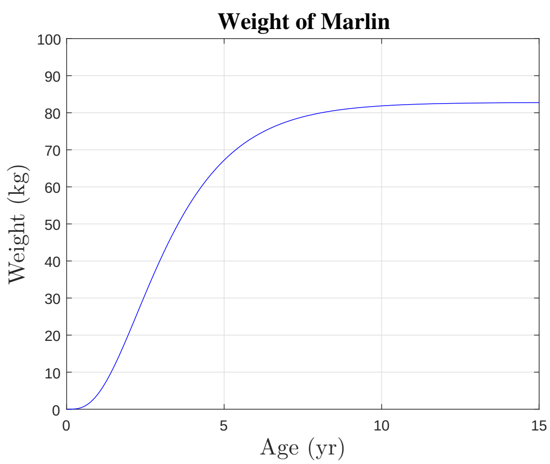
$$W(t) = a\left(L_\infty(1 - e^-bt)\right)^k.$$

f. The functions above are combined in a composite function to give $W(t)$ for the Striped Marlin,

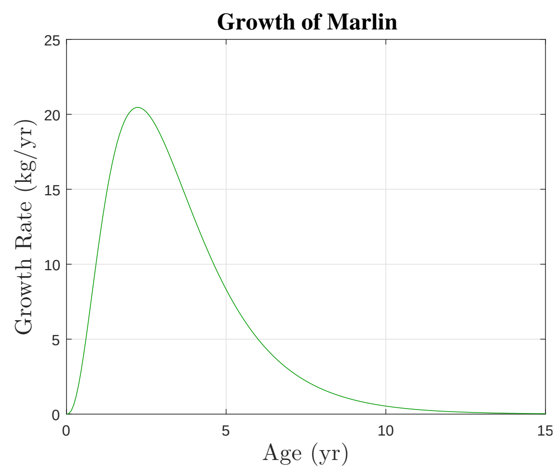
$$W(t) = 82.7983\left(1 - e^{-0.5764t}\right)^{3.6166}.$$

The growth function satisfies the derivative of $W(t)$ with

$$W'(t) = 172.6020184\left(1 - e^{-0.5764\,t}\right)^{2.6166} e^{-0.5764\,t}.$$




$W(t)$




$W'(t)$

The graph of $W(t)$ shows the increase in weight as the fish ages with the increase accelerating for the first 2-3 years then slowing down as it approaches a maximum weight for large time. The point of inflection for the first graph matches the maximum of the growth curve on the right. The growth curve shows the increasing growth rate until approximately the age of 2.25 before growth slows to almost zero for older Marlin. This maximum growth shows the 2 year old Marlin putting on almost 20 kg/yr.

11. The graphs and discussions for this problem are found with the written solutions to this problem. Here the techniques for solving the three different differential equations are shown.

a. We consider the linear DE with initial condition (IVP):

$$\frac{dA}{dt} + kA = be^{-qt}, \qquad A(0) = 0 \quad \text{and} \quad k \neq q.$$

It is clear that the integrating factor is $\mu(t) = e^{kt}$, so

$$\frac{d}{dt}\left(e^{kt}A\right) = be^{(k-q)t}, \qquad \text{so} \qquad e^{kt}A(t) = \frac{be^{(k-q)t}}{k-q} + C.$$

This is readily solved to give:

$$A(t) = \frac{b}{k-q} e^{-qt} + Ce^{-kt}.$$

The initial condition gives $C = -\frac{b}{k-q}$, so

$$A(t) = \frac{b}{k-q} \left( e^{-qt} - e^{-kt} \right).$$

c. The accumulation of lead is the simplest DE, requiring only integration. The DE satisfies:

$$\frac{dP}{dt} = \frac{Kb}{k-q} \left( e^{-qt} - e^{-kt} \right), \qquad \text{with} \quad P(0) = 0.$$

Upon integration we have

$$P(t) = \frac{Kb}{k-q} \left( \frac{e^{-kt}}{k} - \frac{e^{-qt}}{q} \right) + C,$$

where the initial condition gives $C = \frac{Kb}{k-q} \left( \frac{1}{q} - \frac{1}{k} \right)$. It follows that the solution satisfies:

$$P(t) = \frac{Kb}{k-q} \left( \frac{e^{-kt}}{k} - \frac{e^{-qt}}{q} + \frac{1}{q} - \frac{1}{k} \right).$$

e. With the linear ODE describing the weight of the boy is written:

$$\frac{dw}{dt} = -r(w - w_\infty) \qquad \text{with} \qquad w(0) = w_0.$$

Make the substitution $z(t) = w(t) - w_\infty$, where $z(0) = w_0 - w_\infty$. This creates the easier problem:

$$\frac{dz}{dt} = -rz \qquad \text{with} \qquad z(0) = w_0 - w_\infty.$$

The solution is

$$z(t) = w(t) - w_\infty = (w_0 - w_\infty) e^{-rt}.$$

It follows that

$$w(t) = w_\infty + (w_0 - w_\infty) e^{-rt}.$$