Spring 2020 Math 531 Name ____________________

Lecture Notes 9: Fourier Transforms B

Note: For full credit you must show intermediate steps in your calculations.

1. (5pts) The Fourier transform pair is defined by:

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega \qquad\qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx,$$

The Fourier transform table (and HW 10.3.6) give the following Fourier transform pair:

$$f(x) = \begin{cases} 0, & |x| > a, \\ 1, & |x| < a, \end{cases} \qquad\qquad F(\omega) = \frac{\sin(a\omega)}{\pi\omega}$$

Use Euler's formula and the function $F(\omega)$ to reduce the Fourier expression of this $f(x)$ to an integral formula on $[0, \infty)$ with its integrand being just sine and cosine functions of x and ω with the parameter a. Explain why this integral produces the real-valued function of x.
(Slide 3)

2. (6pts) The result in Problem 1 is an integral, which cannot be integrated by elementary methods. Use the numerical integrator in a program of your choosing to evaluate this integral for $a = 2$, using $\omega \in [-10, 10]$, $[-50, 50]$ and $[-100, 100]$. Show the graphic output of these results for $x \in [-10, 10]$, displaying the original $f(x)$ and the approximations. Be sure to include your program. (Show separately the graphs of the 3 ranges of the *frequency spectrum* overlaying $f(x)$.) You might enjoy playing with your program over different ranges of the spectrum, including non-symmetric ones to see how this affects your graph.
(Slide 3)

3. (5pts) On Slide 20 we have the result of the *Convolution Theorem*, which effectively gives us the *Green's function* for our infinite domain *Heat equation*. The formula is given by:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \sqrt{\frac{\pi}{kt}} e^{-(x-s)^2/4kt} \, ds.$$

Use this result to create a heat surface for the following heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad -\infty < x < \infty,$$

with initial condition:

$$u(x,0) = \begin{cases} 2 - |x|, & |x| < 2, \\ 0, & |x| > 2. \end{cases}$$

Use some integration program (like the ones provided) to produce your graphical output, and show your program. For the numerical integration, take your spectral limits to be $s \in [-50, 50]$. For this infinite dimensional surface, show $x \in [-5, 5]$ and $t \in [0, 10]$. Be sure to label all axes.
(Slides 20-23).