Spring 2020 Math 531 Name ______________________________

Lecture Notes 5: Laplace's Equation on a Sphere

Note: For full credit you must show intermediate steps in your calculations.

1. (6pts) Consider the *Laplace's equation in a sphere* with no θ dependence:

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) = 0.$$

with the *boundary condition* $u(a, \phi) = F(\phi)$. Find the general solution to this problem, showing all steps using separation of variables and including the Fourier coefficients (Slides 35-41).

2. (4pts) Suppose that the boundary condition for Problem 1 satisfies:

$$u(a, \phi) = F(\phi) = \alpha \cos(2\phi) + \beta \cos(\phi).$$

Write $F(\phi)$ as a sum of *Legendre polynomials* in $\cos(\phi)$, then use orthogonality to simplify your answer in Problem 1 and write this fairly simple form of $u(\rho, \phi)$. (Slides 30, 35-41).

3. (6pts) Use the one-dimensional Rayleigh quotient (Slide 4 of Lecture Notes: Sturm-Liouville Problem: Part C) to show that $\mu \geq 0$ (if $m \geq 0$) for the 2^{nd} Sturm-Liouville problem (in ϕ) listed on Slide 24. Are there conditions that give $\mu = 0$ as an eigenvalue, and if so list those conditions?