Spring 2020 Math 531 Name ______________________________

Lecture Notes 4: Wave Equation on a Sphere

Note: For full credit you must show intermediate steps in your calculations.

1. (4pts) In the lecture notes we showed the *power series* solution for *Legendre's equation*:

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) + n(n+1)g = 0.$$

Use the *ratio test* to show that our series solutions converge for all $x \in (-1,1)$. In addition, prove that this series diverges for $x = \pm 1$ unless n is an integer (Slides 27-28).

2. (5pts) Use *Rodrigues' formula* to find $P_6(x)$, then use the formula on Slide 32 to find the *associated Legendre polynomial*, $P_6^2(x)$ (Slides 30-32). Graph these two functions for $x \in (-1,1)$.

3. (7pts) a. Consider the Sturm-Liouville problem with Dirichlet BCs for shells inside a sphere:

$$\frac{d}{d\rho}\left(\rho^2\frac{du}{d\rho}\right) + \lambda\rho^2 u = 0, \qquad 1 < \rho < 4,$$
$$u(1) = 0, \qquad u(4) = 0.$$

This is the *spherical Bessel function* with no ϕ-dependence. To solve this ordinary differential equation, you may find it useful to let $u(\rho) = v(\rho)/\rho$ and first solve the equation for $v(\rho)$ (though the case $\lambda = 0$ should be fairly easy to solve). Find the eigenvalues and eigenfunctions, and state the orthogonality relationship (Slide 33).

b. Let $\phi_n(\rho)$ be the eigenfunctions in Part a. Find the generalized Fourier coefficients b_n for

$$f(\rho) = \frac{1}{\rho} = \sum_{n=1}^{\infty} b_n\phi_n(\rho).$$