Spring 2020 Math 531 Name ______________________________

**Lecture Notes 1: Higher Dimensional PDEs**

**Note:** For full credit you must show intermediate steps in your calculations.

1. (5pts) Consider the heat equation:

$$\frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

with the BCs:

$$u(x,0,t) = 0, \qquad u(x,H,t) = 0,$$
$$u_x(0,y,t) = 0, \qquad u_x(L,y,t) = 0,$$

and IC

$$u(x,y,0) = \alpha(x,y).$$

Repeat the steps done for the wave equation for this heat equation, emphasizing the changes caused by the insulated BCs at $x = 0$ and $x = L$. You are not asked to create any 3D pictures for this case. Give the Sturm-Liouville problems, the product solution, the superposition solution, and the Fourier coefficients. (Slides 4-10)

2. (3pts) Slide 10 takes the solution given from the superposition principle:

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( a_{mn} \cos\left(c\sqrt{\lambda_{mn}}t\right) + b_{mn} \sin\left(c\sqrt{\lambda_{mn}}t\right)\right) \sin\left(\tfrac{m\pi x}{L}\right) \sin\left(\tfrac{n\pi y}{H}\right),$$

and states the **orthogonality** gives the Fourier coefficients:

$$a_{mn} = \frac{4}{LH} \int_0^H \int_0^L \alpha(x,y) \sin\left(\tfrac{m\pi x}{L}\right) \sin\left(\tfrac{n\pi y}{H}\right) dx\, dy.$$

Provide the details on how this expression is derived. That is, you need to state clearly some of the intermediate steps on this slide.

3. (4pts) Let $L = \nabla^2$ be a linear operator. Show that this operator is *self-adjoint* with the BCs of Problem 1 above. That is, suppose that

$$\nabla^2 \phi_i + \lambda_i \phi_i = 0, \qquad \text{for} \quad i = 1,2,$$

with boundary conditions:

$$\phi_i(x,0) = 0, \quad \phi_i(x,H) = 0, \quad \frac{\partial \phi_i}{\partial x}(0,y) = 0, \quad \frac{\partial \phi_i}{\partial x}(L,y) = 0, \qquad \text{for} \quad i = 1,2,$$

then

$$\iint_R \left(\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1\right) dR = \iint_R \left(\phi_1 L[\phi_2] - \phi_2 L[\phi_1]\right) dR = 0.$$

(Slide 20-21)

4. (4pts) Consider the set of functions:

$$\phi_1(x) = 1, \quad \phi_2(x) = x, \quad \phi_3(x) = x^2, \quad \phi_4(x) = x^3.$$

Define the inner product of two functions $u(x)$ and $v(x)$ as

$$\langle u, v \rangle = \int_{-1}^{1} u(x)v(x)dx.$$

The two functions, $u(x)$ and $v(x)$, are orthogonal if $\langle u, v \rangle = 0$. Use the Gram-Schmidt process (Slides 24-26) to find a set of mutually orthogonal functions, $\psi_1$, $\psi_2$, $\psi_3$, and $\psi_4$, based on the functions, $\phi_1$, $\phi_2$, $\phi_3$, and $\phi_4$.