Spring 2020 Math 531 Name ______________________

Lecture Notes 10: Fourier Transforms C

Note: For full credit you must show intermediate steps in your calculations.

1. (6pts) The *Fourier sine and cosine transform pairs* are defined by:

$$f(x) = \int_0^\infty F(\omega)\sin(\omega x)d\omega \qquad \mathcal{S}[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)dx$$

$$f(x) = \int_0^\infty F(\omega)\cos(\omega x)d\omega \qquad \mathcal{C}[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)dx$$

Consider $f(x) = e^{-\alpha x}$, $\alpha > 0$ $(x \geq 0)$. Use the definitions of $\mathcal{S}[f(x)]$ and $\mathcal{C}[f(x)]$ to find each of these Fourier transforms.
(Slide 6)

b. With our function $f(x) = e^{-\alpha x}$, $\alpha > 0$ $(x \geq 0)$ and the results computed in Part a, verify the differentiation rules for *Fourier sine and cosine transforms*. That is, with $f(x) = e^{-\alpha x}$ show that:

$$\mathcal{C}\left[\tfrac{df}{dx}\right] = -\frac{2}{\pi}f(0) + \omega\mathcal{S}[f] \qquad \mathcal{S}\left[\tfrac{df}{dx}\right] = -\omega\mathcal{C}[f],$$

and for the second derivatives:

$$\mathcal{C}\left[\tfrac{d^2f}{dx^2}\right] = -\frac{2}{\pi}\tfrac{df}{dx}(0) - \omega^2\mathcal{C}[f] \qquad \mathcal{S}\left[\tfrac{d^2f}{dx^2}\right] = \frac{2}{\pi}\omega f(0) - \omega^2\mathcal{S}[f].$$

(Slide 8)

2. (10pts) Find the solution for Laplace's equation in the first quadrant

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x > 0, \quad y > 0,$$

with the boundary conditions:

$$\frac{\partial}{\partial x}u(0,y) = 0, \qquad u(x,0) = \begin{cases} 10, & 0 < x < 5 \\ 0, & x \geq 5 \end{cases}.$$

Show all steps in the separation of variables, including the appropriate selection of functions that keep $u(x,y)$ tending to zero as x and $y \to \infty$. Write how you obtain your Fourier coefficients and evaluate those integrals. Write your solution in either a sum or integral form that can be readily programmed.
(Slides 20-37)

b. Use your solution in Part a to create a 3D plot of $u(x,y)$ with $x \in [0,20]$ and $y \in [0,20]$. Your program should use at least 100 terms in any Fourier series and integrate at least $\omega \in [0,100]$ for Fourier transforms. Be sure to include your program.
(Slides 20-37)