I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

1. a. Find the eigenvalues and eigenfunctions for the Sturm-Liouville problem

$$\begin{aligned} &\phi'' + \lambda\phi = 0, && 0 < x < 6, \\ \text{with B.C.'s} \quad & 2\phi(0) - \phi'(0) = 0 \quad \text{and} && \phi'(6) = 0. \end{aligned}$$

b. Use the eigenfunctions from Part a to represent the function

$$f(x) = \begin{cases} 0, & 0 < x < 2, \\ 10, & 2 \le x < 6. \end{cases}$$

and find the generalized Fourier coefficients.

c. What does the Fourier series converge to at $x = 1$? at $x = 2$? at $x = 5$? at $x = 0$? at $x = 8$? Does this Fourier series produce a periodic extension for all x? Explain.

d. Use the computer to find the numerical values of the first 50 eigenvalues. (Only write the values for λ_1, λ_2, λ_5, λ_{10}, λ_{25}, and λ_{50}.) Graphically, show $f(x)$ and the approximation using 50 terms in the Fourier series for $x \in [0, 6]$ and $x \in [-6, 12]$. What is the absolute error between your 50 term Fourier series and the value of $f(x)$ at $x = 0.5$, $x = 1.8$, $x = 2.2$, and $x = 5.5$. With your 50 term Fourier series approximation of $f(x)$, give both the x_{min} and $x_{max} \in (0, 6)$ values and the Fourier series value at x_{min} and x_{max} (absolute minimum and maximum values of the Fourier series for $x \in [0, 6]$). Find the maximum actual error between the 50 term approximation and the actual function.

2. A better model for the string problem is given by the nonhomogeneous partial differential equation:

$$u_{tt} + 2ku_t = c^2 u_{xx} - g, \qquad t > 0 \quad \text{and} \quad 0 < x < 1,$$

where k is a small positive constant ($k \ll c\pi$), which accounts for air resistance, and g is the acceleration due to gravity on the string. Assume that the ends of the string are fixed with $u(0, t) = 0$ and $u(1, t) = 0$.

a. Find the equilibrium position for the string, $u_E(x)$.

b. Let $w(x, t) = u(x, t) - u_E(x)$ and show that $w(x, t)$ satisfies a linear homogeneous partial differential equation. Solve this problem when the initial displacement is the same as the equilibrium position, $u(x, 0) = u_E(x)$, and the initial velocity is 1 at each point of the string, *i.e.*, $u_t(x, 0) = 1$. Find $u(x, t)$ and determine the limit of $u(x, t)$ as $t \to \infty$.

3. Consider the heat equation given by:

$$\frac{\partial u}{\partial t} = k\nabla^2 u, \quad 0 < x < 6, \quad 0 < y < 4, \quad t > 0.$$

With boundary conditions:

$$\frac{\partial u}{\partial x}(0, y, t) = 0, \qquad \frac{\partial u}{\partial x}(6, y, t) = 0,$$

and

$$\frac{\partial u}{\partial y}(x, 0, t) = Ax^2, \qquad \frac{\partial u}{\partial y}(x, 4, t) = \begin{cases} x, & 0 \le x < 3, \\ 6 - x, & 3 \le x \le 6, \end{cases}$$

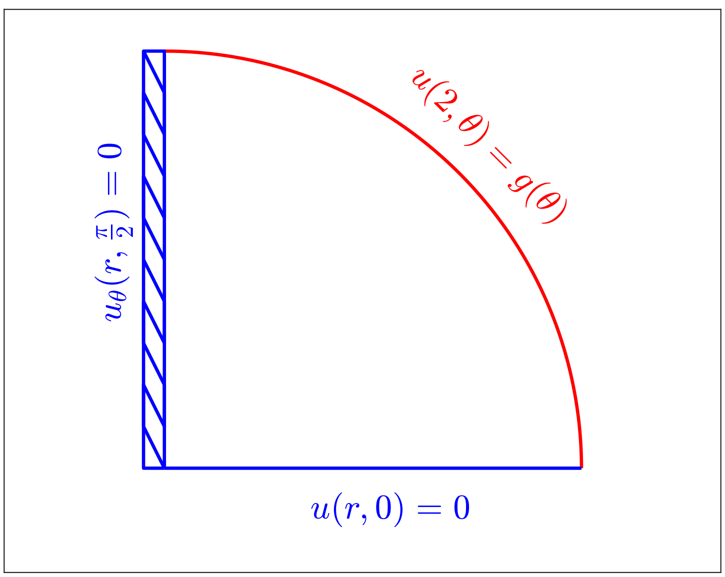
and initial condition:

$$u(x, y, 0) = (6 - x)y.$$

Find the condition on A (A constant) that allows the steady state problem to be solvable on the rectangular domain. Solve the steady state problem.

4. a. Find the steady-state temperature distribution for the Figure below (assuming the faces are insulated). The region is a quarter-circular region satisfying Laplace's equation, where the edge along the positive x-axis is fixed at 0 and the edge along the y-axis is insulated. Along the quarter-circular edge, we have:

$$u(2, \theta) = g(\theta) = \begin{cases} \theta, & 0 \le x < \frac{\pi}{4}, \\ \frac{\pi}{2} - \theta, & \frac{\pi}{4} \le x \le \frac{\pi}{2}. \end{cases}$$



b. Create a colored heat map displaying the steady-state temperature distribution in this region. Include your program.

5. If convection is taken into account, the equation for heat conduction and convection in a one-dimensional rod is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}, \quad 0 < x < L, \quad t > 0.$$

Let $k = 1$, $v_0 = 0.4$, and $L = 8$. Assume the following boundary conditions and initial conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

a. Use separation of variables to create two ordinary differential equations.

b. From the spatial ordinary differential equation, create a Sturm-Liouville eigenvalue problem. Identify explicitly the functions $p(x)$, $q(x)$, and $\sigma(x)$. Find the eigenvalues and eigenfunctions for this problem. Explicitly write the orthogonality condition for this problem.

c. Solve the original partial differential equation with its boundary and initial conditions. Write clearly your integral for finding the Fourier coefficients.

6. a. Consider the 4^{th} order linear operator:

$$L = \frac{d^4}{dx^4},$$

with the boundary conditions:

$$\phi(0) = 0, \quad \phi''(0) = 0, \quad \phi(6) = 0, \quad \text{and} \quad \phi''(6) = 0.$$

Show that L is self-adjoint.

b. With the operator L and boundary conditions in Part a, consider the eigenvalue problem:

$$L[\phi] = \lambda\phi. \tag{1}$$

Prove that the eigenvalues are not complex. Multiplying (1) by ϕ and integrating from $x = 0$ to 6, we have something related to the Rayleigh-Quotient:

$$\lambda = \frac{\int_0^6 \phi L[\phi]dx}{\int_0^6 \phi^2 dx}.$$

Use this (with integration properties) to prove the eigenvalues satisfy $\lambda > 0$. Determine the eigenfunctions and prove that distinct eigenfunctions are orthogonal.

c. The displacement of a uniform thin beam in a medium that resists motion satisfies the beam equation:

$$\frac{\partial^4 u}{\partial x^4} = -\frac{\partial^2 u}{\partial t^2} - 0.2\frac{\partial u}{\partial t}, \qquad 0 < x < 6, \quad t > 0.$$

If the beam is simply supported at the ends, then the boundary conditions are:

$$u(0,t) = 0, \qquad u_{xx}(0,t) = 0, \qquad u(6,t) = 0, \qquad u_{xx}(6,t) = 0.$$

Assume that for the beam there is initially no displacement, $u(x,0) = 0$, and that an initial velocity satisfies:

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 0, & x \in (0,1), \\ 2, & x \in (1,2), \\ 0, & x \in (2,6). \end{cases}$$

Solve this initial-boundary value problem.

d. Use 50 terms in the series solution of $u(x, t)$ and have the computer graph the displacement of the beam at times $t = 0, 1, 2, 5, 10$, and 20. In addition, use MatLab to create a smooth surface showing the time evolution of the beam for $0 \leq x \leq 6$ and $0 \leq t \leq 20$. You must include your programs.