Heat Equation and Fourier Transforms

We showed that $e^{-i\omega x} e^{-k\omega^2 t}$ solve the heat equation, $u_t = ku_{xx}$, so

$$u(x,t) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega.$$  

The IC is satisfied if:

$$f(x) = \int_{-\infty}^{\infty} c(\omega) e^{-i\omega x} d\omega.$$  

From the definition of the Fourier transform, the above equation is a Fourier integral representation of $f(x)$ with

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx.$$  

The Fourier coefficient can be inserted into the solution:

$$u(x,t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{i\omega s} ds \right] e^{-i\omega x} e^{-k\omega^2 t} d\omega.$$  

Interchanging the order of integration gives:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left[ \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega(x-s)} d\omega \right] ds.$$  

However, the inverse Fourier transform of $e^{-k\omega^2 t}$

$$g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega = \sqrt{\frac{\pi}{kt}} e^{-x^2/4kt}.$$
Heat Equation and Fourier Transforms

Fourier Transforms of Derivatives

Fundamental Solution and $\delta(x)$

Example

We insert the information above into the solution and obtain:

$$u(x,t) = \int_{-\infty}^{\infty} f(s) \left[ \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} \right] ds.$$  

It follows that each initial temperature “influences” the temperature at time $t$ according to the Influence function, which is related to the Green’s functions last section:

$$G(x,t;s,0) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}}. $$

This Influence function has problems near $t = 0$.

Dirac Delta function, $\delta(x)$

Define the function:

$$f(x,a) = \begin{cases} 
0, & |x| > a, \\
\frac{1}{2a}, & |x| < a.
\end{cases}$$

The Dirac delta function satisfies:

$$\lim_{a \to 0} f(x,a) = \delta(x).$$

With regards to our Heat problem, we see that as $t \to 0$ the influence is concentrated locally:

$$\lim_{t \to 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} = \delta(x-s).$$

Fundamental Solution

Suppose all the heat is concentrated at the origin, $u(x,0) = \delta(x)$, then

$$u(x,t) = \int_{-\infty}^{\infty} \delta(s) \left[ \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} \right] ds = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.$$

Example: Consider the infinite rod satisfying the heat equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty,$$

with IC

$$u(x,0) = f(x) = \begin{cases} 
0, & x < 0, \\
100, & x > 0.
\end{cases}$$

From above the solution satisfies:

$$u(x,t) = \int_{-\infty}^{\infty} f(s) \left[ \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} \right] ds,$$

$$= \frac{100}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-s)^2}{4kt}} ds.$$
Heat Equation and Fourier Transforms

With the change of dummy variables in the integral, $z = (s - x)/\sqrt{4kt}$, the solution can be written:

$$u(x, t) = \frac{100}{\sqrt{\pi}} \int_{0}^{\infty} e^{-(s-\cdot)^2/4kt} ds,$$

$$= \frac{100}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz,$$

$$= \frac{100}{\sqrt{\pi}} \left[ \int_{0}^{\infty} e^{-z^2} dz + \int_{0}^{\infty} e^{-z^2} dz \right],$$

by the evenness of $e^{-z^2}$.

Thus, we can write the solution:

$$u(x, t) = 50 + \frac{100}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2} dz,$$

$$= 50 \left(1 + \text{erf} \left( \frac{x}{\sqrt{4kt}} \right) \right).$$

Below is the MatLab code for the previous figures for Heat Propagation.

```
1 % Solutions to the heat flow equation
2 % on one-dimensional rod
3 % Fourier Transform solution
4 format compact;
5 tfin = 100; % final time
6 xwid = 10;
7 k = 1; % heat capacitance
8 NptsT=151; % number of t pts
9 NptsX=151; % number of x pts
10 t=linspace(0,tfin,NptsT);
11 x=linspace(-xwid,xwid,NptsX);
12 [T,X]=meshgrid(t,x);
13 clf
14 U = 50*(1 + erf(X./(sqrt(4*k*T)))); % Temperature(n)
15 set(gca,'FontSize',[12]);
16 surf(T,X,U);
17 shading interp
18 colormap(jet)
19 xlabel('$t$','Fontsize',12,'interpreter','latex');
20 ylabel('$x$','Fontsize',12,'interpreter','latex');
21 zlabel('$u(x,t)$','Fontsize',12,'interpreter','latex');
22 axis tight
23 view([30 12])
24 print -dpng heatFT1.png
25 print -depsc heatFT1.eps
```

The temperature spreads by diffusion.

The thermal energy spreads with infinite propagation speed.
Again consider the **Heat equation**:
\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad -\infty < x < \infty,
\]
with **IC**, \(u(x,0) = f(x)\).

**Separation of variables** motivated the **Fourier transform**.

Now solve this directly with **Fourier transform**.

Define
\[
\mathcal{F}[u] = U(\omega, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t)e^{i\omega x} \, dx
\]
be the **Fourier transform** of \(u(x,t)\).

Similarly, **Fourier transforms** of higher derivatives may be obtained:
\[
\mathcal{F} \left[ \frac{\partial^2 u}{\partial x^2} \right] = -i\omega \mathcal{F} \left[ \frac{\partial u}{\partial x} \right] = (-i\omega)^2 U(\omega, t) = -\omega^2 U(\omega, t).
\]

In general, the **Fourier transform** of the \(n^{th}\) derivative of a function with respect to \(x\) equals \((-i\omega)^n\) time the **Fourier transform** of the function, assuming that \(u(x,t) \to 0\) sufficiently fast as \(x \to \pm \infty\).

From the properties of the **Fourier transforms** of the derivatives, the **Fourier transform** of the heat equation becomes:
\[
\frac{\partial}{\partial t} U(\omega, t) = -k\omega^2 U(\omega, t).
\]
Fourier Transforms of Derivatives

The **Fourier transform** acting on the temperature function, \( u(x,t) \), converts the linear partial differential equation with constant coefficients into an ordinary differential equation, since the spatial derivatives are transformed into algebraic multiples of the transform.

Since

\[
\frac{\partial}{\partial t} \mathcal{U}(\omega,t) = -k\omega^2 \mathcal{U}(\omega,t),
\]

the solution becomes

\[
\mathcal{U}(\omega,t) = c(\omega)e^{-k\omega^2 t},
\]

where the arbitrary constant may depend on \( \omega \).

The function \( c(\omega) \) comes from the **IC**, \( f(x) \), so

\[
c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} \, dx,
\]

which gives the same result as obtained by **separation of variables**.

Convolution

Note that

\[
\begin{align*}
    h(x) &= \int_{-\infty}^{\infty} H(\omega)e^{-i\omega x} \, d\omega = \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{-i\omega x} \, d\omega, \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left[ \int_{-\infty}^{\infty} g(s) e^{i\omega s} \, ds \right] e^{-i\omega x} \, d\omega, \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s) \left[ \int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-s)} \, d\omega \right] ds, \\
    h(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)f(x-s) \, ds, \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w)g(x-w) \, dw.
\end{align*}
\]

This is the **convolution** of \( f(x) \) and \( g(x) \) usually denoted

\[
g \ast f = f \ast g
\]

Convolution and Heat Equation

For the **heat equation**, consider the transform \( \mathcal{U}(\omega,t) \) of the solution \( u(x,t) \), where

\[
\mathcal{U}(\omega,t) = c(\omega)e^{-k\omega^2 t}.
\]

- \( c(\omega) \) is the transform of the initial temperature, \( f(x) \).
- \( e^{-k\omega^2 t} \) is the transform of the **fundamental solution**, \( \sqrt{\frac{\pi}{kt}}e^{-x^2/4kt} \).

The **Convolution theorem** gives the solution:

\[
u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \sqrt{\frac{\pi}{kt}}e^{-(x-s)^2/4kt} \, ds.
\]
Enter the Maple commands for the graph of \( u(x,t) \)

\[
\begin{align*}
g := (x,t) \rightarrow (1/(2\pi))\int (\sqrt{\pi/t}) \exp(-1/4*(x-s)^2/t), s = -2 \ldots 2) ; \\
\text{plot3d}(g(x,t), x = -10..10, t = 0.0001..20); \\
\end{align*}
\]

The IC is

\[
f(x) = \begin{cases} 
1, & |x| < 2, \\
0, & |x| > 2. 
\end{cases}
\]

This graph shows the diffusion of the heat with time.

The problem can be done in MatLab using its integral function, which uses an adaptive quadrature to solve the problem.

1. \% Solution Heat Equation with FT 
2. \% Arbitrary f(x) 
3. N1 = 201; N2 = 201; 
4. tv = linspace(0.0001,20,N1); 
5. xv = linspace(-10,10,N2); 
6. [t1,x1] = ndgrid(tv,xv); 
7. f = @(s,c) sqrt(pi/c(1)) *exp(-(c(2)-s).^2/(4 *c(1))); 
8. for i = 1:N1 
9. \hspace{1em} for j = 1:N2 
10. \hspace{2em} c = [t1(i,j),x1(i,j)]; 
11. \hspace{2em} U(i,j) = \ldots 
12. \hspace{2em} (1/(2*pi))*integral(@(s)f(s,c),-2,2); 
13. end 
14. end 
15. 

The Fourier Transform technique for solving PDEs is as follows:

- **Fourier Transform** the PDE in one of the variables, often \( x \).
- Solve the ODE in the other variable, often \( t \).
- Apply the ICs, determining the initial Fourier Transform.
- Use the convolution theorem to obtain the solution.

If the IC is only defined on a finite interval, then often Maple can manage the integral and produce a 3D plot.
Parseval's Identity

Since \( h(x) \) is the inverse of the Fourier Transform of \( F(\omega)G(\omega) \):

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)f(x-s) \, ds = \int_{-\infty}^{\infty} G(\omega)F(\omega)e^{-\omega x} \, d\omega.
\]

Since this holds for all \( x \), it holds for \( x = 0 \), so

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)f(-s) \, ds = \int_{-\infty}^{\infty} G(\omega)F(\omega) \, d\omega.
\]

Take \( g^*(x) = f(-x) \) to be the complex conjugate, then

\[
F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(-s)e^{-i\omega s} \, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} g^*(x)e^{-i\omega x} \, dx = G^*(\omega).
\]

Parseval’s Identity:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)g^*(x) \, dx = \int_{-\infty}^{\infty} G(\omega)G^*(\omega) \, d\omega,
\]

or equivalently,

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |g(x)|^2 \, dx = \int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega.
\]

Energy is often proportional to \( |g(x)|^2 \), so

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |g(x)|^2 \, dx
\]

is the total energy.

The quantity \( |G(\omega)|^2 \) represents the energy per unit wave number, which is the spectral energy density.

The Fourier Transform, \( G(\omega) \), of a function \( g(x) \) is a complex quantity whose magnitude squared is the spectral energy density (or amount of energy per unit wave number).