1. (10 pts) During the human female menstrual cycle, the gonadotropin, FSH or follicle stimulating hormone, is released from the pituitary in a sinusoidal manner with a period of approximately 28 days. Guyton's text on *Medical Physiology* shows that if we define day 0 \((t = 0)\) as the beginning of menstruation, then FSH, \(F(t)\), cycles with a high concentration of about 4 ("relative units") around day 9 and a low concentration of about 1.5 around day 23.

a. Consider a model of the concentration FSH (in "relative units") given by

\[
F(t) = A + B \cos(\omega(t - \phi)),
\]

where \(A, B, \omega,\) and \(\phi\) are constants and \(t\) is in days. Use the data above to find the four parameters, then sketch a graph for the concentration of FSH over one period. If ovulation occurs around day 14, then what is the approximate concentration of FSH at that time?

\[
A = \frac{1.5 + 4}{2} = 2.75, \quad B = 1.25, \quad \omega = \frac{2\pi}{28}, \quad \omega = \frac{\pi}{1.25} = 0.2244
\]

\[
\phi = 0, \quad F(14) = 3.2924
\]

b. Create an equivalent model in the form:

\[
G(t) = C + D \sin(\nu(t - \psi)),
\]

with \(\psi \in [0, T)\), where \(T\) is the period of the function.

\[
C = 2.75, \quad D = 1.25, \quad \nu = 0.2244, \quad \psi = 2
\]

\[
\frac{\pi}{1.25} (9 - \frac{\pi}{2}) = \frac{\pi}{2} \Rightarrow 9 - \phi = 7 \Rightarrow \phi = 2
\]
2. (10 pts) a. *Staphylococcus aureus* is a common cause of skin infections and can lead to serious complications in hospitals, including death (MRSA). A common means of measuring populations of bacteria is through optical density (*OD*$_{650}$). Suppose a culture satisfies the Malthusian growth:

$$ P_{n+1} = (1 + r)P_n, $$

where $n$ is in minutes. If the initial *OD*$_{650}$ is 0.043, i.e., $P_0 = 0.043$, and after 25 min, $P_{25} = 0.071$, then find the value of $r$. Determine the doubling time for this culture and estimate the *OD*$_{650}$ reading at 60 min ($P_{60}$), assuming continued Malthusian growth. (Give all numbers to at least 4 significant figures.)

$$ P_n = 0.043(1 + r)^n, \quad 0.071 = 0.043(1 + r)^{25}, \quad 1 + r = \left(\frac{71}{43}\right)^{\frac{1}{25}} $$

$$ r = \frac{0.020262}{0.020262}, \quad P_{60} = 0.14327 \cdot \text{OD}_{650} $$

Doubling time = 34.555 min

b. A mutant strain also grows according to a Malthusian growth law:

$$ M_{n+1} = (1 + s)M_n. $$

Assume this culture has a doubling time of 31 min and begins with less than 10% of the population, *OD*$_{650}$ is 0.004, or $M_0 = 0.004$. Determine the value of $s$ and find a general expression for $M_n$.

$$ M_n = 0.004(1 + s)^n, \quad z = (1 + s)^{\frac{1}{s}}, \quad 1 + s = z, $$

$$ s = 0.022611, \quad M_n = 0.004(1.022611)^n $$

c. Assuming these cultures start at the same time, find how long it takes for them to have the same *OD*$_{650}$ reading.

$$ P_m = M_m \implies 0.043(1.020262)^m = 0.004(1.022611)^m $$

$$ \left(\frac{43}{4}\right) = \left(\frac{1.022611}{1.020262}\right)^m $$

$$ P_m = M_m, \text{ when } m = 1032.39 \text{ min} $$

$$ m = \frac{\ln\left(\frac{43}{4}\right)}{\ln\left(\frac{1.022611}{1.020262}\right)} $$