May 1, 2017 Math 124a Name Key

Note: For full credit you must show intermediate steps in your calculations. This includes steps for product, quotient, and chain rules for differentiation and substitutions for integrals, which need them.

1. (9pts) Evaluate the following integrals:

a. $\int \left(\frac{1}{2x^3} + 12\cos(4x) \right)\, dx,$ b. $\int \frac{6t\, dt}{\sqrt{t^2+4}}.$ $u = t^2+4$ $du = 2t\, dt$

$\frac{1}{2}x^{-3}$

$3\int u^{-1/2}\, du = 6u^{1/2} + C$

2.5, 2.5
4

a. $-\frac{1}{4}x^{-2} + 3\sin(4x) + C$ b. $6(t^2+4)^{1/2} + C$

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2. (11pts) a. Consider the curves

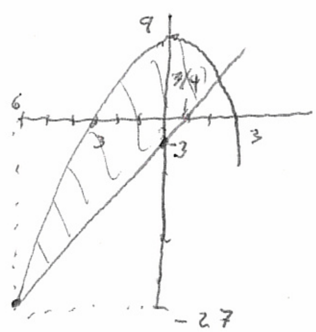
$$y = 4x - 3 \quad \text{and} \quad y = 9 - x^2.$$

Find all points of intersection. Graph the curves, showing all x and y-intercepts. Set up (show limits and integrand) and solve the integral that determines the closed area above the line and below the parabola. Shade the area of interest.

$4x - 3 = 9 - x^2 \Rightarrow x^2 + 4x - 12 = (x+6)(x-2) = 0$

2 Points of Intersection: $(-6, -27), (2, 5)$

3 **SKETCH of the AREA**



3 Integral Defining Area: $\int_{-6}^{2} [(9-x^2) - (4x-3)]\, dx$ dx

$= \int_{-6}^{2} [12 - 4x - x^2]\, dx = 12x - 2x^2 - \frac{x^3}{3} \Big|_{-6}^{2}$

3 Area = $\frac{256}{3}$

$= (24 - 8 - \frac{8}{3}) + 72 + 72 - 72$

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3. (25pts) a. A study of crickets, *Diestrammena asynamora*, with a limited food supply, best fits a logistic growth model of the form:

$$P_{n+1} = F(P_n) = P_n + 1.25\,P_n\left(1 - \frac{P_n}{80.6}\right), \qquad P_0 = 10,$$

where n is in weeks. Compute the derivative of $F(P)$. Find all equilibria. Determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function. (Credit for stability and behavior will be given only if information to the left of the circled quantities is correct and behavior analyses match appropriate equilibrium.)

$$F'(P) = 1 + 1.25 - \frac{2.5P}{80.6}$$

2 $F'(P) = $ $2.25 - \frac{2.5P}{80.6} = 2.25 - 0.031017P$

3 $P_{1e} = $ 0 $F'(P_{1e}) = $ 2.25 **Stable** or **Unstable** **Monotonic** or **Oscillatory**

3 $P_{2e} = $ 80.6 $F'(P_{2e}) = $ -0.25 **Stable** or **Unstable** **Monotonic** or **Oscillatory**

b. An alternate model that is popular for insects is Hassell's model. The best fitting version of this model for the experiment above is given by:

$$P_{n+1} = H(P_n) = \frac{2.78\,P_n}{(1 + 0.005\,P_n)^3}, \qquad P_0 = 10.$$

Compute the derivative of $H(P)$. Determine the maximum of this function (both P and $H(P)$ values). Sketch a graph of $H(P)$ for $P \geq 0$ along with the identity map ($P_{n+1} = P_n$) and give the points of intersection. (Next page.)

$$H'(P) = 2.78\,\frac{(1+0.005P)^3 - 3P(1+0.005P)^2(0.005)}{(1+0.005P)^6} = 2.78\,\frac{(1-0.01P)}{(1+0.005P)^4}$$

3 $H'(P) = $ $2.78\,\frac{(1-0.01P)}{(1+0.005P)^4}$

3 Maximum $P_c = $ 100 and $H(P_c) = $ 82.3704

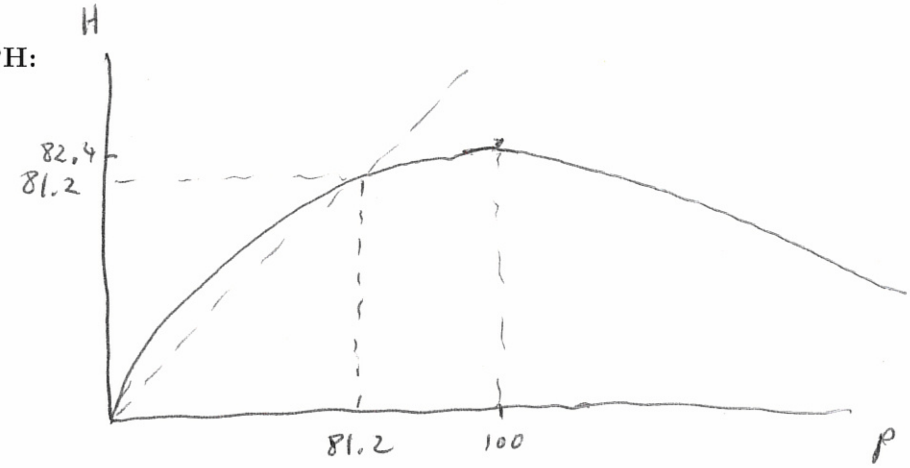
2 Points of Intersection of $H(P_n)$ with $P_{n+1} = P_n$ $(0,0)$, $(81.219, 81.219)$

$$P_e = \frac{2.78P_e}{(1+0.005P_e)^3} \Rightarrow P_e = 0 \text{ or } (1+0.005P_e)^3 = 2.78$$

$$P_e = 200\,(2.78^{1/3} - 1)$$

2

GRAPH:



c. Find all equilibria. Determine the stability of the equilibria. Justify your stability argument by evaluating the derivative of the updating function. (Credit for stability and behavior will be given only if information to the left of the circled quantities is correct and behavior analyses match appropriate equilibrium.)

3 $P_{1e} =$ 0 $H'(P_{1e}) =$ 2.78 **Stable** or **Unstable** **Monotonic** or **Oscillatory**

4 $P_{2e} =$ 81.219 $R'(P_{2e}) =$ 0.13357 **Stable** or **Unstable** **Monotonic** or **Oscillatory**

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4. (6pts) Evaluate the following definite integral:

$$\int_0^4 \frac{6t\,dt}{(t^2+2)^2}.$$

$u = t^2 + 2$ $du = 2t\,dt$

6 $= 3\int_2^{18} u^{-2}\,du = -3u^{-1}\Big|_2^{18} = -\frac{1}{6} + \frac{3}{2} = \frac{4}{3}$

Definite integral = 1.3333

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5. (12pts) Solve the following initial value problems:

a. $\frac{dy}{dt} = 6 - \frac{y}{3}$, $\quad y(0) = 8$,

$= -\frac{1}{3}(y-18)$

$z(t) = y(t) - 18$

$z(0) = 8 - 18 = -10$

$z' = -\frac{1}{3} z \qquad z(t) = -10e^{-t/3} = y(t) - 18$

5

$y(t) =$ $18 - 10e^{-t/3}$

b. $\frac{dy}{dt} = 4t\sin(t^2 + \pi)y^{1/2}$, $\quad y(0) = 25$,

$\int y^{-1/2}\,dy = \int 4t\sin(t^2+\pi)\,dt \qquad u = t^2 + \pi, \quad du = 2t\,dt$

$y(0) = 25 = (\frac{c}{2} + 1)^2 \Rightarrow \frac{c}{2} = 4$

$2y^{1/2} = 2\int \sin(u)\,du = -2\cos(u) + c$

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$y^{1/2} = \frac{c}{2} - \cos(t^2+\pi) \Rightarrow y(t) = (\frac{c}{2} - \cos(t^2+\pi))^2$

$y(t) =$ $(4 - \cos(t^2+\pi))^2$

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6. (5pts) Assume that a reasonable model for the population of Mexico over a limited period of time is the Malthusian growth model:

$$\frac{dP}{dt} = rP.$$

The population of Mexico was 52.78 million in 1970 and expanded to 99.93 million in 2000. Use the model above to find the growth rate, r, and find how long for the population to double according to this model. $t = 0$ at 1970

$P(t) = 52.78e^{rt} \qquad P(30) = 99.93 = 52.78e^{30r}$

$r = \frac{1}{30}\ln\left(\frac{99.93}{52.78}\right)$

5

$r =$ 0.021278 $\quad$ Doubling time = 32.576 yr

$t_d = \frac{\ln(2)}{r}$

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7. (9pts) a. A lake maintains a constant volume of $V = 500,000$ m^3 of water. An initial monitoring for a pollutant finds a concentration $c(0) = 2\,\mu$g/m^3. There are two streams entering this lake with differing concentrations of the pollutant. The first stream has a flow rate of $f_1 = 600$ m^3/day with a pollutant concentration of $Q_1 = 12\,\mu$g/m^3. A second stream has a flow rate of $f_2 = 400$ m^3/day with the pollutant concentration of $Q_2 = 4\,\mu$g/m^3. Assume that this is a well-mixed lake with a stream flowing out at a rate of $f_3 = 1000$ m^3/day. Write a differential equation describing the concentration of pesticide in the lake ($c(t)$) and solve this differential equation. Also, find how long until the lake has a concentration of $5\,\mu$g/m^3 of pollutant.

$$\frac{dA}{dt} = \left(\overbrace{600\cdot 12 + 400\cdot 4}^{8800} - 1000\cdot c\right)\frac{1}{V}$$

$z(t) = c(t) - 8.8$

$z(0) = 2 - 8.8 = -6.8$

$z' = -0.002z$

$z(t) = -6.8e^{-0.002t}$

3 $\quad \dfrac{dc}{dt} = \dfrac{88}{5000} - \dfrac{10}{5000}c = -0.002(c - 8.8)$

$5 = 8.8 - 6.8e^{-0.002t}$

4 $\quad c(t) = 8.8 - 6.8e^{-0.002t}$

$e^{0.002t} = \dfrac{6.8}{3.8}$

2 $\quad c(t_2) = 5$ when $t_2 =$ 290.96 day

$t = 500\ln\left(\dfrac{6.8}{3.8}\right)$

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8. (8pts) a. A tissue culture is placed at a specified distance from an ^{131}I, sample for a dose response experiment. Measurements show the tissue culture is receiving a radiation dose from this decaying source, and the cumulative exposure over 7 days satisfies:

$$\int_0^7 7.3e^{-0.0845t}dt. = -\frac{7.3}{0.0845}e^{-0.0845t}\Big|_0^7$$

$$= 86.39\left(1 - e^{-0.0845\cdot 7}\right) = 38.574$$

5 $\quad$ Exposure = 38.574 mCi

b. Find how long the tissue culture would need to be at this distance from the ^{131}I source if you wanted its total exposure to be 60 mCi, *i.e.*, find T such that

$$\int_0^T 7.3e^{-0.0845t}dt = 60.$$

$$86.3905\left(1 - e^{-0.0845T}\right) = 60 \Rightarrow 26.3905 = 86.3905e^{-0.0845T}$$

$$e^{0.0845T} = \frac{86.3905}{26.3905}$$

3 $\quad T =$ 14.034 days

$$T = \frac{1}{0.0845}\ln\left(\frac{86.3905}{26.3905}\right)$$

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