March 22, 2017 Math 124a Name Key

1. (13pts) Differentiate the following functions, showing clearly the product rule, quotient rule, or chain rule when applicable (**DON'T simplify**):

a. $g(x) = e^{-3x}\ln(x^4+5) + \frac{1}{4x} + \frac{\sin(x^2+2)}{x^3+8}$.

5, 3, 5 $g'(x) = e^{-3x} \cdot \frac{4x^3}{x^4+5} - 3e^{-3x}\ln(x^4+5) - \frac{1}{4}x^{-2} + \frac{(x^3+8)2x\cos(x^2+2) - 3x^2\sin(x^2+2)}{(x^3+8)^2}$

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2. (13pts) Draw the graph of the following function:

$$y = \frac{0.5(x^2-3)}{x+2}.$$

Find the y and x-intercepts. Find any asymptotes (vertical and/or horizontal). Give the derivative of the function, then determine extrema, (local maxima and/or minima, including the x and y values). Sketch this function. (**If a maximum or minimum or particular asymptote doesn't exist, then write DOESN'T APPLY or DNA.**)

1, 2 y-intercept = $-\frac{3}{4}$ x-intercept(s) = $\pm\sqrt{3} = \pm 1.7321$

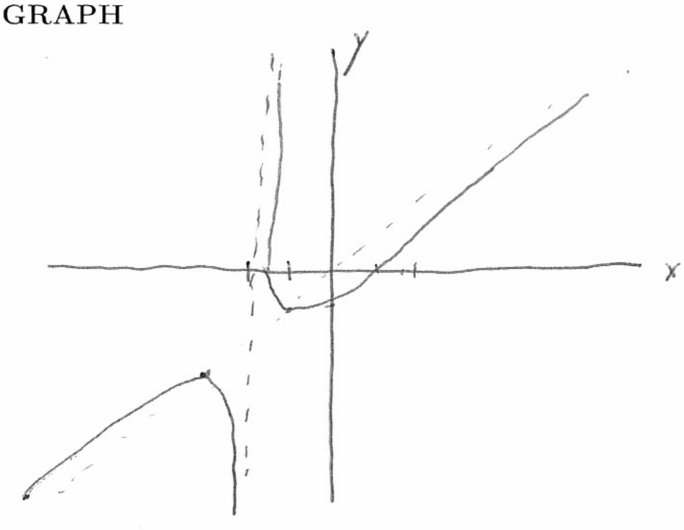
1, 1 Horizontal Asymptote y = DNA Vertical Asymptote x = -2

3 $y'(x) = 0.5\,\frac{(x+2)2x - (x^2-3)(1)}{(x+2)^2} = 0.5\,\frac{x^2+4x+3}{(x+2)^2} = \frac{0.5(x+3)(x+1)}{(x+2)^2}$

1.5, 1.5 x_{max} = -3 $y(x_{max})$ = -3 x_{min} = -1 $y(x_{min})$ = -1

Sketch of GRAPH

2



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3. (13pts) Gompertz developed a model for the growth of a tumor. A particular type of tumor growing according to Gompertz's model satisfies the growth law,

$$G(N) = 0.1N(4 - 0.6\ln(N)) \quad \text{(thousand cells/day)},$$

where N is the number of tumor cells (in thousands) and the time units are days. Note that this model is not defined for $N = 0$, and has been shown to work poorly for very low populations of tumor cells.

a. Find all equilibria for this Gompertz model by solving $G(N) = 0$.

$G(N_e) = 0 \Rightarrow 0.6\ln(N_e) = 4 \Rightarrow \ln(N_e) = 20/3 \qquad N_e = e^{20/3}$

2 $N_e =$ 785.77 (thousand)

b. Compute the derivative of $G(N)$ and find the maximum of the function (both N and $G(N)$ values).

$G'(N) = 0.1\left(N\left(-\frac{0.6}{N}\right) + (4 - 0.6\ln(N))\right) = 0.1(3.4 - 0.6\ln(N))$

$\ln(N_{max}) = \frac{17}{3} \qquad N_{max} = e^{17/3}$

3 $G'(N) =$ $0.1(3.4 - 0.6\ln(N))$

1, 1 $N_{max} =$ 289.07 (thousand) $\qquad G(N_{max}) =$ 17.344 (thousand/day)

c. Find the value for the growth function for $N = 400$ (thousand cells). Also find the rate of the growth function of the tumor for this size. Describe whether the tumor is growing or decreasing according to your results, and whether the velocity of this growth/decrease is increasing or decreasing.

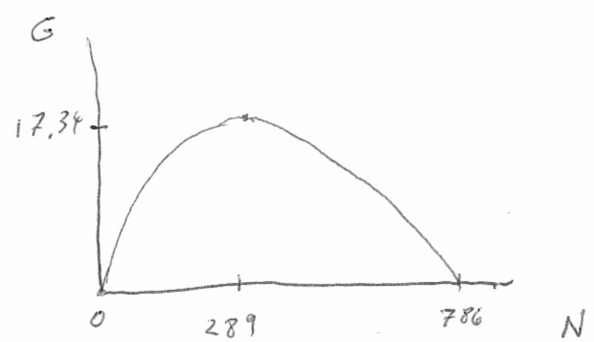
1, 1 $G(400) =$ 16.205 $\qquad G'(400) =$ − 0.01949

1 For $N = 400$: The tumor is (circle one:) **growing** **shrinking**

1 The rate of growth is (circle one:) **increasing** **decreasing.**

d. Sketch a graph of $G(N)$. Clearly label your axes and include scales on them.

2



4. (18pts) a. Oxytocin (OT) is mammalian neurohypophysial hormone, which is known to promote pair bonding. It is a short-lived hormone, which binds to key neural receptors to encourage "trust and empathy." It is well-documented to be released during child birth, breastfeeding, and orgasm. The body maintains a basal level, then its concentration rises after breastfeeding (thought to be an evolutionary response to promote this behavior). Below is a model, which fits data for a female subject who begins nursing an infant at $t = 0$ for about 8 min. Suppose that the concentration of OT, $O(t)$, is given by the function,

$$O(t) = 2.3 + 30\left(e^{-0.075t} - e^{-0.16t}\right),$$

where t is in minutes and $[O]$ is in ng/ml. Find the derivative $O'(t)$. Find when oxytocin achieves its maximum concentration and determine what its maximum concentration is.

$0.16e^{-0.16t} = 0.075e^{-0.075t} \Rightarrow \frac{160}{75} = e^{0.085t}$

1 O-intercept = 2.3

2 Horizontal Asymptote O = 2.3

3 $O'(t) = 30\left(0.16e^{-0.16t} - 0.075e^{-0.075t}\right)$

2, 1 $t_{max} = 8.9139$ $O(t_{max}) = 10.467$

b. Another scientist believes that an alternate model fits the data better. This model satisfies the equation:

$$Q(t) = 2.4 + 2.6t\,e^{-0.11t},$$

where again t is in minutes and $[Q]$ is in ng/ml. Find the derivative $Q'(t)$. Find when oxytocin achieves its maximum concentration and determine what its maximum concentration is.

$2.6\left(t(-0.11)e^{-0.11t} + e^{-0.11t}\right)$ $t_{max} = \frac{1}{0.11}$

1 Q-intercept = 2.4

2 Horizontal Asymptote Q = 2.4

3 $Q'(t) = 2.6e^{-0.11t}(1 - 0.11t)$

2, 1 $t_{max} = 9.0909$ $Q(t_{max}) = 11.095$

5. (17pts) a. A pronghorn antelope is startled by a cougar and must clear a 5.2 ft fence to escape. Assume that the pronghorn antelope jumps the fence with just enough vertical velocity, v_0 to clear it. If the height (in ft) of the pronghorn antelope is given by:

$$h(t) = -16\,t^2 + v_0\,t, \qquad v(t) = -32t + v_0$$

where h is the height in ft and t is in sec. Use the height of the fence (maximum height) and the function describing the height of the pronghorn antelope, $h(t)$, to determine the vertical velocity, v_0, then determine how long the pronghorn antelope is in the air.

$t_0 = \frac{v_0}{32} \quad h\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + \frac{v_0^2}{32} = \frac{v_0^2}{64} = 5.2$

3,1 $\quad v_0 =$ 18.243 ft/sec $\qquad$ Time in air = 1.1402 sec

b. When the pronghorn antelope sees the cougar, it experiences a rush of adrenalin, which helps power its muscles to escape the predator. This powerful hormone is very important in animals for escaping predation. Its production in the body is tightly regulated through a negative feedback control mechanism, which leads to concentrations in the blood resembling a damped oscillator. Suppose that concentration of adrenalin in this antelope satisfies the equation:

$$A(t) = 11 + 310\,e^{-0.36t}\sin(0.28t), \qquad t \geq 0,$$

where t is in min after sighting the cougar and $A(t)$ is in ng/l of blood. Determine $A'(t)$, then find the absolute maximum and minimum concentrations of adrenalin in the blood ($t \geq 0$) and when these extremes occur.

$0.28\cos(0.28t) = 0.36\sin(0.28t) \Rightarrow \tan(0.28t) = \frac{28}{36}$

$t_m = \frac{1}{0.28}\arctan\left(\frac{7}{9}\right) = 2.3609$

3 $\quad A'(t) = 310e^{-0.36t}\left(0.28\cos(0.28t) - 0.36\sin(0.28t)\right)$

2,1 $\quad$ Absolute Maximum $t_{max} =$ 2.3609 min $\qquad A(t_{max}) =$ 92.353 ng/l

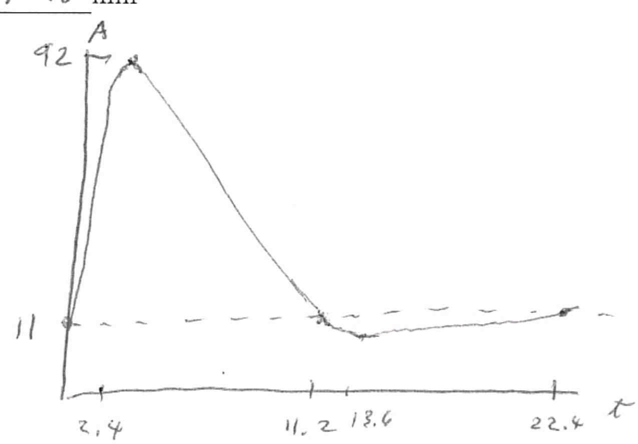
2,1 $\quad$ Absolute Minimum $t_{min} =$ 13.581 min $\qquad A(t_{min}) =$ 9.567 ng/l

c. Find the first **3** times that $A(t_i) = 11$ for $t_i \geq 0$. Sketch a graph of $A(t)$. Clearly label your axes and include scales on them.

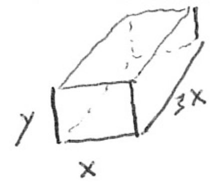
2 $\quad t_i =$ 0, 11.220, 22.440 min

$t_0 = \frac{\pi}{0.28} = 11.220$

2



6. (12pts) A closed rectangular box must have 7500 cm^3 of space, and its base must have its length be three times its width. Find the dimensions of this box, which is constructed with the least amount of building material possible. Clearly present your formulas for volume and surface area. Also, show your steps for deriving the optimal solution.



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Objective: $S(x,y) = 6x^2 + 8xy$

Constraint: $V(x,y) = 3x^2y = 7500$

$$y = \frac{7500}{3x^2} = \frac{2500}{x^2}$$

$$S(x) = 6x^2 + \frac{20000}{x}$$

4

$$S'(x) = 12x - \frac{20000}{x^2} = 0$$

$$x^3 = \frac{20000}{12} \Rightarrow x = \left(\frac{5000}{3}\right)^{1/3} = 11.856$$

Length = 35.569 cm Width = 11.856 cm Height = 17.784 cm

4

Surface Area = 2530.30 cm^2