1. Find the area of the largest rectangle with a base on the x-axis and the upper vertices on the parabola $y = 12 - x^2$. Give the dimensions of this rectangle.

2. A rectangular study plot is bounded on one side by a river, and the other three sides are to be blocked off by a fence. Find the dimensions of the plot that maximizes the area enclosed with 20 meters of fence.

3. An open box with its base having a length twice its width is to be constructed with 600 in^2 of material. Find its dimensions that maximize the volume.

4. Find the dimensions of an open rectangular box with a square base that holds 32 in^3 and is constructed with the least building material possible.

5. Find the dimensions of a right circular cylindrical can with both a top and a bottom that holds one liter (1000 cm^3) and is constructed with the least amount of material possible.

6. The strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from a circular log with a radius of 25 cm.

7. Nutrients in low concentrations inhibit growth of an organism, but high concentrations are often toxic. Let c be the concentration of a particular nutrient (in moles/liter) and P be the population density of an organism (in number/cm^2). Suppose that it is found that the effect of this nutrient causes the population to grow according to the equation:

$$P(c) = \frac{1000c}{1 + 100c^2}.$$

Find the concentration of the nutrient that yields the largest population density of this organism and what the population density of this organism is at this optimal concentration.

8. One question for fishery management is how to control fishing to optimize profits for the fishermen. We will soon study the continuous logistic growth equation for populations. One differential equation describing the population dynamics for a population of fish F with harvesting is given by the equation,

$$\frac{dF}{dt} = rF\left(1 - \frac{F}{K}\right) - xF,$$

where r is the growth rate of this species of fish at low density, K is the carrying capacity of this population, and x is the harvesting effort of the fishermen. We will show that the non-zero equilibrium of this equation is given by

$$F_e = K\frac{(r - x)}{r}.$$

One formula for profitability is computed by the equation

$$P = xF_e,$$

so

$$P(x) = Kx\frac{(r-x)}{r}.$$

Find the level of harvesting x that produces the maximum profit possible with this dynamics, where x_{max} might depend on any of the parameters listed in the problem. What is the equilibrium population, F_e, at this optimal profitability? Also, determine the maximum possible fish population for this model and at what harvesting level this occurs. (Clearly, this is a grossly oversimplified model, but can give some estimates for long range management.)

9. (From [1]) Semelparous organisms breed only once during their lifetime. Examples of this type of reproduction strategy can be found with Pacific salmon and bamboo. The per capita rate of increase, r, can be thought of as a measure of reproductive fitness. The greater r, the more offspring an individual produces. The intrinsic rate of increase is typically a function of age, x. Models for age-structured populations of semelparous organisms predict that the intrinsic rate of increase as a function of x is given by

$$r(x) = \frac{\ln[L(x)M(x)]}{x},$$

where $L(x)$ is the probability of surviving to age x and $M(x)$ is the number of female births at age x.

Suppose that

$$L(x) = e^{-0.21x}$$

and

$$M(x) = 6x^{1.1}.$$

Find the optimal age of reproduction.

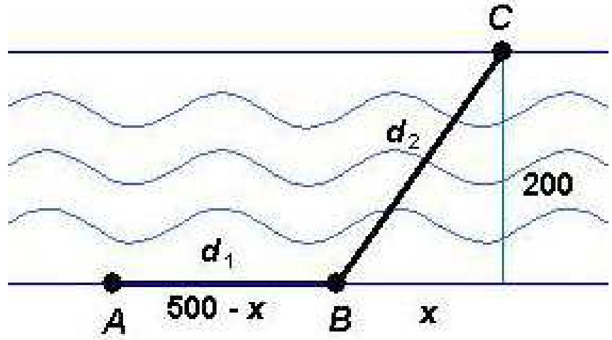
[1] D. A. Roff, *The Evolution of Life Histories*, Chapman & Hall, 1992.

10. A female otter hears the cries of distress from her young in a den across and up the river from where she is foraging. (See the diagram below.) Assume that she is initially at Point A with the den residing at Point C. She wants to reach her young in the minimum amount of time. Assume she can run along the bank at $v_1 = 10$ ft/sec and swim through the river as $v_2 = 6$ ft/sec. The river is 200 ft wide and the den is 500 ft up the river. (We are ignoring the current in the river.) If the distance she runs along the bank (from A to B) is d_1 and the distance she swims (from B to C) is d_2, then the time for her to reach the den is given by the formula

$$T = \frac{d_1}{v_1} + \frac{d_2}{v_2}.$$

a. Use the diagram below to form an expression for the time as a function of x (the distance downstream from the den, where she crosses), $T(x)$.

b. Use your expression for the time $T(x)$ to find the minimum time for the otter to reach her pups. Give both the distance x and the time at the minimum.

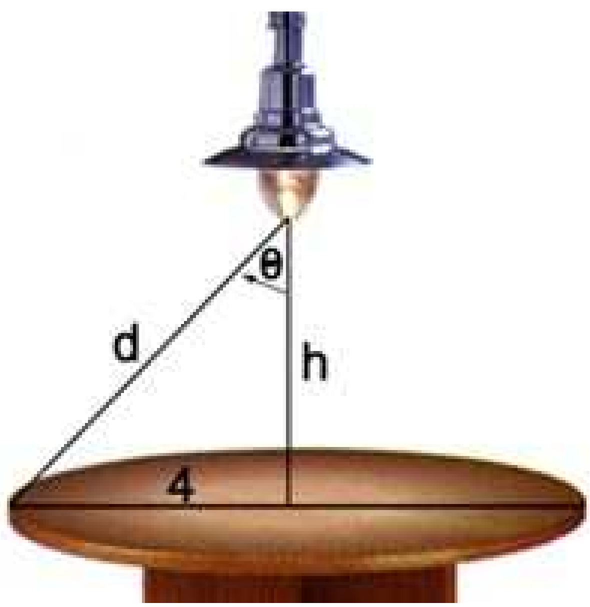


11. A lamp with an adjustable height hangs above a circular table with a radius of 4 ft. A right triangle is formed between the center of the table, the lamp, and the edge of the table. The height h is the edge of the triangle between the center and the lamp, and the hypotenuse, d, is the distance from the lamp to the edge. Suppose that the illumination of the edge of the table I is directly proportional to the cosine of the angle θ, which is formed between the edge h and the hypotenuse d of the triangle, and is inversely proportional to the square of the distance d between the lamp and the edge of the table, so

$$I = \frac{3.4\cos(\theta)}{d^2}.$$

a. Find the distance d in terms of $\sin(\theta)$. With this information, eliminate the d from the expression for I, giving $I(\theta)$. Find the derivative, $I'(\theta)$.

b. How close to the table should the lamp be to maximize the illumination at the edge of the table?



12. In a beehive, each cell is a regular hexagonal prism (with radius and side length of R), open at one end with a trihedral angle at the other end. The trihedral end consists of 3 identical

rhombuses that meet at a vertex with the face of these rhombuses forming an angle θ with the vertical of the cell. Measurements of these honeycomb cells have indicated that bees minimize the surface area of these cells for a fixed volume, V (to hold the pupae). From the geometry of the cells, it can be shown that the total surface area S is given by the formula:

$$S(\theta) = \frac{4V\sqrt{3}}{3R} - \frac{3R^2}{2}\frac{\cos(\theta)}{\sin(\theta)} + \frac{3R^2\sqrt{3}}{2}\frac{1}{\sin(\theta)}.$$

a. Calculate the derivative $\frac{dS}{d\theta}$. Your answer may include the parameters R and V. You may want to use the trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to simplify your expression.

b. Find the $\cos(\theta)$ that minimizes the surface area.