1. For the initial value problem, $\frac{dy}{dt} = 2y, \quad y(0) = 6$, the solution is given by

$$y(t) = 6e^{2t}.$$

2. We rewrite the linear differential equation $\frac{dz}{dt} = 0.1z - 2 = 0.1(z - 20)$. We make the substitution $w(t) = z(t) - 20$ with $\frac{dw}{dt} = \frac{dz}{dt}$ and $w(0) = z(0) - 20 = -15$. The reduced problem becomes

$$\frac{dw}{dt} = 0.1w, \qquad w(0) = -15.$$

which has the solution $w(t) = -15e^{0.1t} = z(t) - 20$. Thus, the solution is

$$z(t) = 20 - 15e^{0.1t}.$$

3. For the initial value problem, $\frac{dx}{dt} = -\frac{x}{3}, \quad x(0) = 10$, the solution is given by

$$x(t) = 10e^{-\frac{t}{3}}.$$

4. For the initial value problem, $\frac{dy}{dt} = 0.02y, \quad y(2) = 50$, the solution is given by

$$y(t) = 50e^{0.02(t-2)}.$$

5. We rewrite the linear differential equation $\frac{dr}{dt} = 1 - \frac{r}{4} = -\frac{1}{4}(r-4)$. We make the substitution $z(t) = r(t) - 4$ with $\frac{dz}{dt} = \frac{dr}{dt}$ and $z(1) = r(1) - 4 = 2$. The reduced problem becomes

$$\frac{dz}{dt} = -\frac{z}{4}, \qquad z(1) = 2.$$

which has the solution $z(t) = 2e^{-\frac{(t-1)}{4}} = r(t) - 4$. Thus, the solution is

$$r(t) = 4 + 2e^{-\frac{(t-1)}{4}}.$$

6. a. A yeast population model satisfies $\frac{dY}{dt} = 0.14Y, \quad Y(0) = 100$. This has the solution:

$$Y(t) = 100e^{0.14t}.$$

The population doubles when $Y(t) = 200$, so $100e^{0.14t_d} = 200$ or $t_d = \frac{\ln(2)}{0.14} \approx 4.95105$ hr.

b. A competing population satisfies the initial value problem, $\frac{dP}{dt} = -0.07P, \quad P(0) = 1000$. This has the solution:

$$P(t) = 1000e^{-0.07t}.$$

This population is halved when $P(t) = 1000e^{-0.07t_h} = 500$ or $e^{0.07t_h} = 2$. It follows that $t_h = \frac{\ln(2)}{0.07} \approx 9.9021$ hr.

c. The populations are equal when $100e^{0.14t} = 1000e^{-0.07t}$ or $e^{(0.14+0.07)t} = 10$. It follows that $t = \frac{\ln(10)}{0.21} \approx 10.9647$ hr.

7. a. Malthusian growth for the population of Canada satisfies the differential equation $C'(t) = k_1C(t), \quad C(0) = 24,070,000$, which by letting $t = 0$ correspond to 1980 has the solution

$$C(t) = 24,070,000e^{k_1 t}.$$

From the population in 1990 ($t = 10$), we have $C(10) = 26,620,000 = 24,070,000e^{10k_1}$ or $e^{10k_1} = \frac{26,620,000}{24,070,000} \approx 1.105941$. It follows that $k_1 \approx 0.1\ln(1.105941) \approx 0.0100697\text{yr}^{-1}$. To find doubling time t_d, we solve $C(t_d) = 2(24,070,000) = 24,070,000e^{k_1 t_d}$ or $e^{k_1 t_d} = 2$. It follows that $t_d = \frac{1}{k_1}\ln(2) \approx 68.835$ yr.

b. A similar argument gives the population of Kenya, $K(t) = 16,681,000e^{k_2 t}$. We solve for k_2 in a similar manner, so $e^{10k_2} = \frac{24,229,000}{16,681,000} \approx 1.45249$. It follows that $k_2 \approx 0.1\ln(1.45249) \approx 0.037328\text{yr}^{-1}$. The doubling time for Kenya is computed similarly with $t_d = \frac{1}{k_2}\ln(2) \approx 18.569$ yr.

c. The populations in 2000 ($t = 20$) are given by $C(20) = 24,070,000e^{20k_1} = 29,440,150$ or $K(20) = 16,681,000e^{20k_2} = 35,192,401$. The populations are equal when $C(t) = K(t)$ or $24,070,000e^{k_1 t} = 16,681,000e^{k_2 t}$. Thus, $e^{(k_2-k_1)t} = \frac{24,070}{16,681} \approx 1.44296$ or $t \approx \frac{1}{(k_2-k_1)}\ln(1.44296) \approx 13.453$ yr. This would be around the middle of 1993.

8. a. This radioactive decay problem satisfies $S' = -kS, \quad S(0) = 20$, which has the solution

$$S(t) = 20e^{-kt}.$$

Since the half-life is 28 years, we have $S(28) = 10 = 20e^{-28k}$ or $e^{28k} = 2$. It follows that $k = \frac{\ln(2)}{28} \approx 0.0247553\ \text{yr}^{-1}$. After 10 years, $S(10) = 20e^{-10k} = 15.6142$ mg.

b. For 7 mg remaining, $S(t) = 20e^{-kt} = 7$ or $e^{kt} = \frac{20}{7}$. It follows that $t = \frac{1}{k}\ln\left(\frac{20}{7}\right) \approx 42.408$ yr.

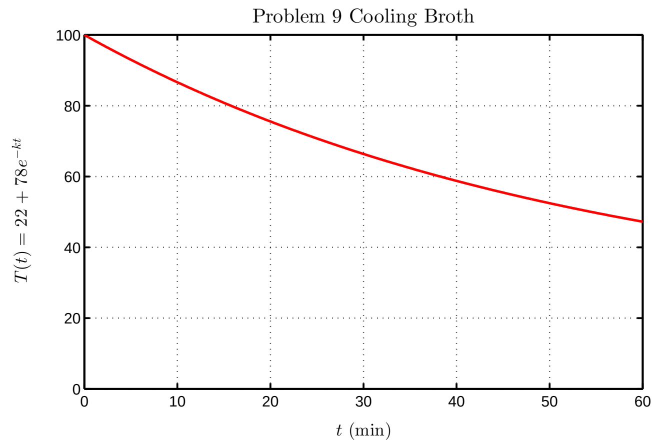
9. a. Newton's Law of Cooling applies to this broth and gives the initial value problem $T(t) = k(T(t)22), \quad T(0) = 100$. We make the substitution $z(t) = T(t) - 22$ with $z'(t) = T(t)$ and $z(0) = T(0) - 22 = 78$. This leaves the reduced problem $z' = -kz, \quad z(0) = 78$, which has the solution $z(t) = 78e^{-kt} = T(t) - 22$. Thus,

$$T(t) = 22 + 78e^{-kt}.$$

Since the broth is 93°C after 5 min, we have $T(5) = 93 = 22 + 78e^{k5}$ or $e^{5k} = \frac{78}{71}$. It follows that $k = \frac{1}{5}\ln\left(\frac{78}{71}\right) \approx 0.0188058$. Thus, the temperature of the broth is

$$T(t) = 22 + 78e^{-0.018806t}$$

b. The time to inoculation satisfies $T(t) = 40 = 22 + 78e^{-0.018806t}$ or $e^{0.018806t} = \frac{78}{18}$. It follows that the time to cool to the correct temperature is $t = \frac{1}{0.018806}\ln\left(\frac{78}{18}\right) \approx 77.97$ min. Thus, you can go to exercise for an hour, as long as you don't linger in the shower afterwards. A sketch of the graph is below.



10. a. Newton's Law of Cooling applies to the thin plate and gives the initial value problem $T\,(t) = k(T(t)20), \quad T(0) = 100$. We make the substitution $z(t) = T(t) - 20$ with $z'(t) = T\,(t)$ and $z(0) = T(0) - 20 = 80$. This leaves the reduced problem $z' = -kz, \quad z(0) = 80$, which has the solution $z(t) = 80e^{-kt} = T(t) - 20$. Thus,

$$T(t) = 20 + 80e^{-kt}.$$

Since the thin plate is 80°C after 10 min, we have $T(10) = 80 = 20 + 80e^{10k}$ or $e^{10k} = \frac{80}{60}$. It follows that $k = 0.1\ln\left(\frac{80}{60}\right) \approx 0.028768$. Thus, the temperature of the thin plate is

$$T(t) = 20 + 80e^{-0.028768t}$$

b. When the thin plate reaches $30°C$, we solve $T(t) = 30 = 20 + 80e^{-kt}$ or $e^{kt} = 8$. Thus, $t = \frac{\ln(8)}{k} \approx 72.283$ min.

11. a. The Malthusian growth model is $P'(t) = kP(t), \quad P(0) = 1500$. This has the solution $P(t) = 1500e^{kt}$. The value of k satisfies $P(4) = 2000 = 1500e^{4k}$ or $e^{4k} = \frac{2000}{1500} = \frac{4}{3}$. It follows that $k = \frac{1}{4}\ln\left(\frac{4}{3}\right) \approx 0.0719205$. The doubling time is $e^{kt_d} = 2$ or $t_d = \frac{\ln(2)}{k} \approx 9.6377$ hr.

b. The initial value problem is given by $P' = -0.1P + 100 = -0.1(P - 1000), \quad P(0) = 5000$. We make the substitution $z(t) = P(t) - 1000$, so $z'(t) = P'(t)$ and $z(0) = P(0) - 1000 = 4000$. The reduced problem is $z' = -0.1z$, which has the solution

$$z(t) = 4000e^{-0.1t} = P(t) - 1000 \quad \text{or} \quad P(t) = 4000e^{-0.1t} + 1000.$$

Since the exponential function decays to zero as $t \to \infty$,

$$\lim_{t\to\infty} P(t) = 1000.$$

12. a. Let $a(t)$ be the amount of pollutant, and the concentration $c(t)$ is the concentration of pollutant (in ppb). The change in amount = the amount entering - the amount leaving. The change in amount, $a(t)$, has units (mass/day). The amount entering is $fQ = 20,000$ ppb·m^3/day

(mass/day), while the amount leaving is $fc(t) = 4000c(t)$ ppb·m^3/day (mass/day). Thus, $a(t) = 20000 - 4000c(t)$, but $c(t) = \frac{a(t)}{V} = \frac{a(t)}{200000}$. It follows that the differential equation for the concentration is:

$$\frac{a'(t)}{200000} = c'(t) = \frac{20000 - 4000c(t)}{200000} = 0.1 - 0.02c(t) = -0.02(c(t) - 5), \quad c(0) = 0.$$

To solve this initial value problem we make the substitution $z(t) = c(t) - 5$, so $z' = -0.02z$, $z(0) = -5$, which yields $z(t) = -5e^{-0.02t} = c(t) - 5$. Thus, the solution is

$$c(t) = 5 - 5e^{-0.02t}.$$

b. We solve $c(t) = 5 - 5e^{-0.02t} = 4$, so $e^{0.02t} = 5$. Thus, $t = 50\ln(5) \approx 80.472$ days. Hence, the concentration reaches 4 ppb around 80.47 days.

c. The exponential function in $c(t)$ decays to zero as $t \to \infty$. Thus, the limiting concentration is $c(t) = 5$ ppb.

d. The pollutants are shut down, so the only change in pollutants is $a'(t) = -4000c(t)$. This leaves the concentration initial value problem $c'(t) = -0.02c(t), \quad c(0) = 5$, which has the solution

$$c(t) = 5e^{-0.02t}.$$

This reaches the level of 4 ppb when $c(t) = 4 = 5e^{-0.02t}$ or $e^{0.02t} = 1.25$. Solving this gives $t = 50\ln(1.25) \approx 11.157$ days.

13. a. Let $a(t)$ be the amount of pollutant, and the concentration $c(t)$ is the concentration of pollutant (in ppb). The change in amount = the amount entering - the amount leaving. The change in amount, $a'(t)$, has units (mass/day). The amount entering is $f_1Q_1 + f_2Q_2 = 4000 \cdot 18 + 2500 \cdot 4 = 82,000$ ppb·m^3/day (mass/day), while the amount leaving is $(f_1 + f_2)c(t) = 6500c(t)$ ppb·m^3/day (mass/day). Thus, the differential equation for the change in amount is

$$\frac{da(t)}{dt} = 82,000 - 6500c(t).$$

The relation between the amount and concentration is $c(t) = \frac{a(t)}{V} = \frac{a(t)}{3000000}$ and $c'(t) = \frac{a(t)}{3000000}$, so the concentration differential equation:

$$\frac{dc(t)}{dt} = \frac{82,000 - 6500c(t)}{3,000,000} - \frac{13}{6000}\left(c - \frac{164}{13}\right) \approx -0.0021667(c - 12.615).$$

With the initial condition $c(0) = 0$, we make the substitution $z(t) = c(t) - 12.615$, so $z(0) = -12.615$ and the differential equation is

$$\frac{dz}{dt} = -0.0021667z, \qquad z(0) = -12.615,$$

which gives $z(t) = -12.615e^{-0.0021667t} = c(t) - 12.615$. Thus,

$$c(t) = 12.615\left(1 - e^{-0.0021667t}\right).$$

b. We solve $c(t) = 12.615\left(1 - e^{-0.0021667t}\right) = 4$, so $e^{0.0021667t} = \frac{12.615}{8.615} = 1.4643$. Thus, $t = \frac{\ln(1.4643)}{0.0021667} \approx 176.01$ days. Hence, the concentration reaches 4 ppb at $t \approx 176.01$ days. The limiting concentration is

$$\lim_{t\to\infty} c(t) = 12.615 \text{ ppb}.$$

This easily follows because the exponential tends to zero for large t.

14. a. According to the von Bertalanffy equation, the fish growth satisfies $\frac{dL}{dt} = k(34 - L(t)) = -k(L(t) - 34), \quad L(0) = 2$. To solve this we make the substitution, $z(t) = L(t) - 34$, which has $z(0) = -32$ and $\frac{dz}{dt} = \frac{dL}{dt}$. The modified differential equation becomes $z' = -kz$, which has the solution $z(t) = -32e^{-kt} = L(t) - 34$. Thus, the length of the fish satisfies:

$$L(t) = 34 - 32e^{-kt}.$$

b. If $L(4) = 10$, then $34 - 32e^{-4k} = 10 \quad \text{or} \quad e^{4k} = \frac{32}{24} = \frac{4}{3}$. It follows that $k = \frac{1}{4}\ln\left(\frac{4}{3}\right) \approx 0.0719205$. The length of the fish satisfies:

$$L(t) = 34 - 32e^{-0.0719205t}.$$

The graph appears below.

c. When $t = 10$, $L(t) = 34 - 32e^{-0.0719205 \cdot 10} \approx 18.4115$ cm. Since the exponential decays to zero as $t \to \infty$, $L(t) \to 34$ cm.