Integrate the following:

1. $$\int \left(4t^3 + 1 - e^{-5t}\right) dt = \frac{4t^4}{4} + t - \frac{e^{-5t}}{-5} + C$$
$$= t^4 + t + \frac{e^{-5t}}{5} + C.$$

2. $$\int \left(2t - 6t^{-2} + \cos(4t)\right) dt = \frac{2t^2}{2} - \frac{6t^{-1}}{-1} + \frac{\sin(4t)}{4} + C$$
$$= t^2 + \frac{6}{t} + \frac{\sin(4t)}{4} + C.$$

1 *a.* $$\int \left(2 - x^{1/2} + 4x^{1/3}\right) dx = 2x - \frac{x^{3/2}}{3/2} + 4\frac{x^{4/3}}{4/3} + C$$
$$= 2x - \frac{2}{3}\sqrt{x^3} + 3\sqrt[3]{x^4} + C.$$

2 *a.* $$\int \left(e^{-x} - 3x^{-1} - x\right) dx = -e^{-x} - 3\ln|x| - \frac{x^2}{2} + C.$$

3. $$\int \left(2x^{-3} - 2x^{-1/2} + e^{4x}\right) dx = 2\frac{x^{-2}}{-2} - 2\frac{x^{1/2}}{1/2} + \frac{e^{4x}}{4} + C$$
$$= -\frac{1}{x^2} - 4\sqrt{x} + \frac{e^{4x}}{4} + C.$$

4. $$\int \left(\frac{1}{5}x^{-1} - 14\sin(7x) + 6x^2\right) dx = \frac{1}{5}\ln|x| + 14\frac{\cos 7x}{7} + 6\frac{x^3}{3} + C$$
$$= \frac{1}{5}\ln|x| + 2\cos(7x) + 2x^3 + C.$$

5. $$\int \left(6t^5 - 4t^{-3} + 2e^{-2t}\right) dt = 6\frac{t^6}{6} - 4\frac{t^{-2}}{-2} + 2\frac{e^{-2t}}{-2} + C$$
$$= t^6 + \frac{2}{t^2} - e^{-2t} + C.$$

6. $$\int \Big((t^3-1)^2 - \cos(2t)\Big)\,dt = \int \Big(t^6 - 2t^3 + 1 - \cos(2t)\Big)\,dt$$

$$= \frac{t^7}{7} - 2\frac{t^4}{4} + t - \frac{\sin(2t)}{2} + C$$

$$= \frac{t^7}{7} - \frac{t^4}{2} + t - \frac{1}{2}\sin(2t) + C.$$

7. For $\frac{dy}{dt} = 2 - 0.1e^{-t}$, we integrate to obtain

$$y(t) = \int \Big(2 - 0.1e^{-t}\Big)\,dt = 2t - 0.1\frac{e^{-t}}{-1} + C = 2t + 0.1e^{-t} + C$$

From the initial condition, $y(0) = 10 = 0 + 0.1 + C$, so $C = 10 - 0.1 = 9.9$.
Thus, the solution is given by

$$y(t) = 2t + 0.1e^{-t} + 9.9.$$

8. For $\frac{dy}{dt} = \frac{2}{t^2}$, we integrate to obtain

$$y(t) = 2\int t^{-2}dt = 2\frac{t^{-1}}{-1} + C = -\frac{2}{t} + C.$$

From the initial condition, $y(1) = 1 = -2 + C$ so $C = 1 + 2 = 3$.
Thus, the solution is given by

$$y(t) = 3 - \frac{2}{t}.$$

9. The equation $\frac{dy}{dt} + 2y = 2$ can be written $\frac{dy}{dt} = -2(y-1)$. This is a linear differential equation, so we let $z(t) = y(t) - 1$. Since $y(0) = 5$, $z(0) = 5 - 1 = 4$. The translated equation is

$$\frac{dz}{dt} = -2z, \qquad z(0) = 4,$$

which has the solution, $z(t) = 4e^{-2t} = y(t) - 1$. So

$$y(t) = 4e^{-2t} + 1.$$

9 a. The equation $\frac{dy}{dt} = 5 - \frac{y}{2}$ can be written $\frac{dy}{dt} = -\frac{1}{2}(y - 10)$. This is a linear differential equation, so we let $z(t) = y(t) - 10$. Since $y(0) = 15$, $z(0) = 15 - 10 = 5$. The translated equation is

$$\frac{dz}{dt} = -\frac{1}{2}z, \qquad z(0) = 5,$$

which has the solution, $z(t) = 5e^{-t/2} = y(t) - 10$. So

$$y(t) = 5e^{-t/2} + 10.$$

10. For $\frac{dy}{dt} = \cos(2t) + 2t$, we integrate to obtain

$$y(t) = \int (\cos(2t) + 2t)\, dt = \frac{\sin(2t)}{2} + 2\frac{t^2}{2} + C = \frac{1}{2}\sin(2t) + t^2 + C.$$

From the initial condition, $y(0) = 4 = C$. Thus, the solution is given by

$$y(t) = \frac{1}{2}\sin(2t) + t^2 + 4.$$

11. For $\frac{dh}{dt} = \frac{2}{t}$, we integrate to obtain

$$h(t) = 2\int t^{-1} dt = 2\ln|t| + C.$$

From the initial condition, $h(1) = 3 = 2\ln|1| + C = C$.
Thus, the solution is given by

$$h(t) = 2\ln|t| + 3.$$

12. a. The differential equation for the velocity is given by

$$\frac{dv}{dt} = -32, \qquad v(0) = 96.$$

Upon integration, this has the solution $v(t) = -32t + C_v$. Since $v(0) = 0 + C_v = 96$,

$$v(t) = 96 - 32t.$$

The differential equation for the height of the ball satisfies

$$\frac{dh}{dt} = v(t) = 96 - 32t, \qquad h(0) = 256.$$

Upon integration, this has the solution

$$h(t) = 96t - 32\frac{t^2}{2} + C_h.$$

From the initial height, $h(0) = 256 = C_h$, so the solution becomes

$$h(t) = 256 + 96t - 16t^2.$$

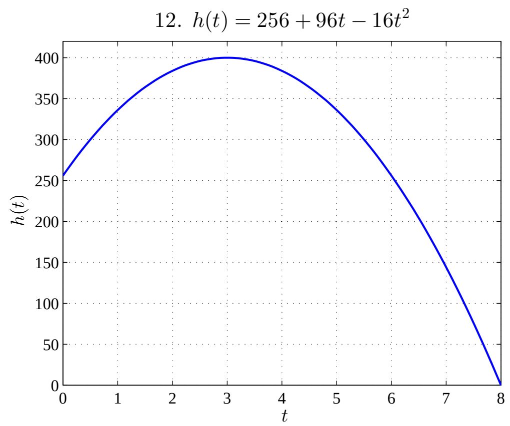
b. At the maximum height, $v(t) = 0$ or $96 - 32t = 0$, which occurs at $t = \frac{96}{32} = 3$ sec. The maximum height satisfies

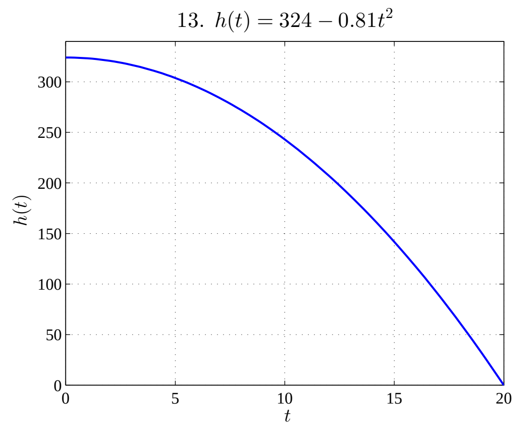
$$h(3) = 256 + 96(3) - 16(9) = 400 \text{ ft}.$$

When it hits the ground, $h(t) = 0$. But

$$h(t) = -16(t^2 - 6t - 16) = -16(t-8)(t+2).$$

Thus, it hits the ground when $t = 8$ sec. (The solution at $t = -2$ doesn't make sense). The velocity of impact is then $v(8) = -32(8) + 96 = -160$ ft/sec. The graph is shown below on the left.





13. a. The differential equation for the velocity is given by

$$\frac{dv}{dt} = -1.62, \qquad \text{so} \qquad v(t) = -1.62t + C_v.$$

With the initial condition, $v(0) = 0$, the solution becomes

$$v(t) = -1.62t.$$

The differential equation for the height is

$$\frac{dh}{dt} = -1.62t \qquad \text{with} \qquad h(0) = 324.$$

Upon integration with the initial condition

$$h(t) = -1.62\frac{t^2}{2} + C_h, \qquad \text{so} \quad h(t) = 324 - 0.81t^2.$$

b. The rock hits the ground when the height $h(t) = 0 = 324 - 0.81t^2$. Thus,

$$t^2 = \frac{324}{0.81} \qquad \text{or} \qquad t = 20 \text{ sec.}$$

The velocity the rocks hits the ground at

$$v(20) = -1.62(20) = -32.4 \text{ m/sec.}$$

The graph is shown above on the right.

14. a. The differential equation for the velocity is given by

$$\frac{dv}{dt} = 9.8, \qquad \text{so} \qquad v(t) = 9.8t + C_v.$$

With $v(0) = 0$, the solution is $v(t) = 9.8t$. The differential equation for the height is

$$v(t) = \frac{dh}{dt} = 9.8t, \qquad \text{so} \qquad h(t) = 4.9t^2 + C_h.$$

With $h(0) = 0$, this gives

$$h(t) = 4.9t^2.$$

When the cat hits the ground, $h(t) = 39.2$ m, so $h(t) = 4.9t^2 = 39.2$ or $t = \sqrt{\frac{39.2}{4.9}} = 2\sqrt{2} \approx 2.828$ sec. It follows that the velocity hitting the ground would be $v(2.828) = 9.8(2.828) = 27.72$ m/s.

b. Actually the air resistance is quite significant, and the differential equation is the linear equation

$$\frac{dv}{dt} = 9.8 - 0.35v = -0.35\,(v - 28)\,.$$

With the substitutions, $z(t) = v(t) - 28$ and $z(0) = v(0) - 28 = -28$, the translated differential equation is

$$\frac{dz}{dt} = -0.35z, \qquad \text{so} \qquad z(t) = -28e^{-0.35t} = v(t) - 28.$$

It follows that

$$v(t) = 28\left(1 - e^{-0.35t}\right).$$

At $t = 3$, we find $v(3) = 28\left(1 - e^{-0.35(3)}\right) \approx 18.202$ m/sec.

c. The position $s(t)$ (with $s(0) = 0$) satisfies the differential equation

$$\frac{ds}{dt} = v(t) = 28 - 28e^{-0.35t}.$$

Thus, integration gives

$$s(t) = \int \left(28 - 28e^{-0.35t}\right) dt = 28t + \frac{28}{0.35}e^{-0.35t} + C = 28t + 80e^{-0.35t} + C.$$

The initial value $s(0) = 0 = 80 + C$ or $C = -80$, which gives the solution

$$s(t) = 28t + 80e^{-0.35t} - 80.$$

It follows that the distance after 3 sec is $s(3) = 28(3) + 80e^{-0.35(3)} - 80 = 31.995$ m.

d. This part of the problem uses Newton's method to find a good approximation of the time and velocity of hitting the ground with this linear model. As noted, Newton's formula is

$$t_{n+1} = t_n - \frac{s(t_n) - 39.2}{v(t_n)}.$$

The initial approximation is $t_0 = 3$, so

$$t_1 = 3 - \frac{31.995 - 39.2}{18.2} = 3.396.$$

Thus, $v(t_1) = v(3.396) = 19.47$ and $s(t_1) = s(3.396) = 39.46$. Hence,

$$t_2 = 3.396 - \frac{39.46 - 39.2}{19.47} = 3.382 \text{ sec.}$$

It follows that $v(t_2) = v(3.382) = 19.43$ m/sec and $s(t_2) = s(3.382) = 39.2$ m.

15. a. The weight of a walleye is given by the differential equation:

$$\frac{dw}{dt} = 0.03(10 - w) = -0.03(w - 10),$$

so we make the substitution $z(t) = w(t) - 10$ with $z(0) = w(0) - 10 = -10$. Thus, the initial value problem becomes

$$\frac{dz}{dt} = -0.03z, \qquad z(0) = -10,$$

which has the solution $z(t) = -10e^{-0.03t} = w(t) - 10$. Thus,

$$w(t) = 10 - 10e^{-0.03t}.$$

The time for a 2 kg walleye satisfies the equation $2 = 10 - 10e^{-0.03t}$ or $8 = 10e^{-0.03t}$. It follows that

$$e^{0.03t} = 1.25 \qquad \text{or} \qquad 0.03t = \ln(1.25).$$

Thus, $t = \frac{100}{3}\ln(1.25) \approx 7.4$ yr.
If you let t get very large, then $w(t) \to 10$ kg, since the exponential decays to zero. The graph appears below on the left.

b. With the result from Part a, the differential equation for the amount of PCBs in the walleye is given by

$$\frac{dP}{dt} = 0.3w(t) = 3 - 3e^{-0.03t}.$$

Thus, by integrating we find $P(t)$, so

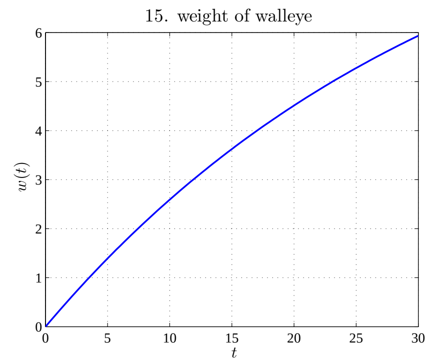
$$P(t) = \int \left(3 - 3e^{-0.03t}\right) dt = 3t - 3\frac{e^{-0.03t}}{-0.03} + C = 3t + 100e^{-0.03t} + C.$$

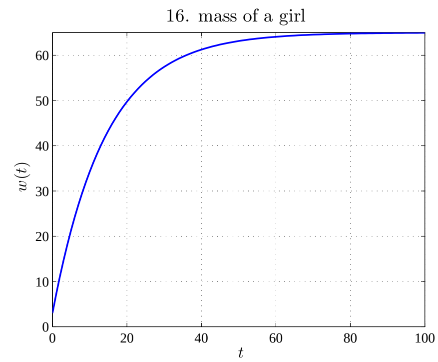
The initial condition gives $P(0) = 0 = 3(0) + 100e^0 + C$ or $C = -100$. Thus, the solution of the differential equation becomes

$$P(t) = 3t + 100e^{-0.03t} - 100.$$

We use this formula to enter the values of $t = 2$, 5, and 10 to get $P(2) = 0.17645$ mg, $P(5) = 1.0708$ mg, and $P(10) = 4.0818$ mg.

c. First substitute into the weight equation from Part a at the times 2, 5, and 10 years to give $w(2) = 0.58236$ kg, $w(5) = 1.3929$ kg, and $w(10) = 2.5918$ kg. Next we use the formula $c(t) = \frac{P(t)}{w(t)}$ with the values for $P(t)$ in Part b to obtain $c(2) = 0.3030$ μg/g, $c(5) = 0.7687$ μg/g, and $c(10) = 1.5749$ μg/g.





16. a. Assume that the mass (in kg) of a girl satisfies the linear differential equation

$$\frac{dw}{dt} = 0.07(65 - w) = -0.07(w - 65), \qquad \text{with} \qquad w(0) = 3,$$

We make the substitution, $z(t) = w(t) - 65$, so $z(0) = w(0) - 65 = 3 - 65 = -62$. The translated differential equation becomes:

$$\frac{dz}{dt} = -0.07z,$$

which has the solution $z(t) = -62e^{-0.07t} = w(t) - 65$. Thus,

$$w(t) = 65 - 62e^{-0.07t}.$$

The predicted masses at times $t = 2$, 4, 7, and 10 are easily computed to give $w(2) = 11.10$ kg, $w(4) = 18.14$ kg, $w(7) = 27.02$ kg, and $w(10) = 34.21$ kg, respectively.

To find the age at which the girl reaches 50 kg, we solve $65 - 62e^{-0.07t} = 50$ or $e^{0.07t} = \frac{62}{15}$. Thus, $t = \frac{100}{7}\ln\left(\frac{62}{15}\right) \approx 20.27$ years. The limiting mass as the girl ages is 65 kg, since the exponential decays to zero. The graph appears above to the right.

b. The differential equation for the amount of lead in the body, $P(t)$ (in μg), satisfies

$$\frac{dP(t)}{dt} = 100e^{-0.4t}w(t) = 100e^{-0.4t}\left(65 - 62e^{-0.07t}\right) = 6500e^{-0.4t} - 6200e^{-0.47t}.$$

This differential equation is integrated to give:

$$P(t) = \int \left(6500e^{-0.4t} - 6200e^{-0.47t}\right) dt = \frac{6500e^{-0.4t}}{-0.4} - \frac{6200e^{-0.47t}}{-0.47} + C$$

With the initial condition, $P(0) = 0$, it follows that

$$\frac{6200}{0.47} - \frac{6500}{0.4} + C = 0 \qquad \text{or} \qquad C = 3058.51.$$

Thus,

$$P(t) = \frac{6200e^{-0.47t}}{0.47} - \frac{6500e^{-0.4t}}{0.4} + 3058.51.$$

To find the amount of lead in a girl at girl ages $t = 2$, 4, 7, and 10, we substitute the values and obtain $P(2) = 909.88$, $P(4) = 1790.58$, $P(7) = 2561.78$, and $P(10) = 2880.86$ μg.

c. The concentration of lead in the blood, c (in μg/dl), satisfies the equation

$$c(t) = \frac{0.1P(t)}{w(t)}.$$

Thus, the concentration of lead in a girl ages 2, 4, 7, and 10 is $c(2) = 8.1972$, $c(4) = 9.8701$, $c(7) = 9.4820$, and $c(10) = 8.4207$ μg/dl.