Spiraling the Earth with C. G. J. Jacobi

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The simple requirement that one should move on the surface of a sphere with constant *speed* while maintaining a constant angular velocity with respect to a fixed diameter, leads to a path whose cylindrical coordinates turn out to be given by the Jacobian elliptic functions. Many properties of these functions can be derived and visualized using this path, known as Seiffert's spiral.

I. INTRODUCTION

Elliptic functions have been known for about 150 years, mainly through the work of the mathematicians Legendre, Abel, and most of all, C. G. J. Jacobi[1] (1804–1851). They are doubly periodic functions of a variable in the complex plane with the additional property that they are meromorphic, i.e., they have only poles as singularities. Among the elliptic functions, the so-called Jacobian elliptic functions are of second order, that is, they have two poles of first order in the period parallelogram.

Most frequently, these latter functions are defined as the inverse functions of certain integrals that cannot be expressed in terms of those functions which, somewhat presumptuously, are called elementary. This definition usually forms the basis of the applications of the Jacobi elliptic functions in physics. In mechanics they appear in the theories of the simple pendulum and of the spinning top (i.e., a rotating rigid body with one point fixed); in superconductivity they appear as the solutions of the Ginzburg–Landau equations; in mathematical biology they occur as solutions of reaction-diffusion equations, etc.

Here I wish to develop an approach to the Jacobian elliptic functions that will appeal to readers who like to think in terms of geometrical images. I believe that this approach will provide an insight into the behavior of these functions for different ranges of their parameters and some understanding of how these functions are interconnected.

This approach was discovered by A. Seiffert and communicated in 1896 in a publication which may have escaped the attention of some of the readers of this Journal.[2] It was briefly referred to by Whittaker and Watson.[3] My contribution to the subject consists only in a few additions and in giving the elliptic functions a pictorial interpretation.

II. AROUND THE WORLD IN ONE DAY

Let us suppose that we start out from above the North Pole in a flying machine at a constant speed $v = ds/dt$, where s is the length of arc traveled in a coordinate system fixed with respect to Earth and t is the time. We define ''circling the globe once'' as the process in which the traveler crosses all meridians once, starting from, but not necessarily returning to, the North Pole. Our aim is to travel in such a manner as to compensate the rotation of the Earth around its axis, so that, roughly speaking, there are always the same stars (or the sun) in the plane passing through the Earth's axis and our position. (In this endeavor we neglect the slow motion of the Earth around the sun.)

This means that, besides the condition $v = \text{constant}$, we also assume

$$\phi = at, \tag{2.1}$$

where ϕ (mod 2π) determines the geographic longitude of our position[4] and $a = 2\pi/\text{day}$ is our *constant angular velocity*.

What kind of curve will our flying machine trace out with respect to the surface of the Earth (considered as a perfect sphere)? The answer is: Seiffert's spherical spiral, whose equation we will now develop.

First, we eliminate the time variable by setting $t = s/v$ and rewrite Eq. (2.1) as

$$\phi = ks, \quad k = \frac{a}{v}. \tag{2.2}$$

Now, k remains the only parameter of our problem. Since we decided that our angular velocity $d\phi/dt$ has to be constant, it follows that if our surface velocity is high, k has to be small, and *vice versa*.

We have to take into account, of course, that the curve in question is located on the surface of a sphere. For simplicity, we assume that our distance from the center of the sphere is unity and that our height above the ground is negligible compared to the sphere's radius. With reference to Fig. 1, we introduce the coordinates of our position P: ϕ is the longitude (see the discussion below) $(0 \leqslant \phi < 2\pi)$; z is the height above the equatorial plane $(-1 \leqslant z \leqslant 1)$; and ρ is the distance (with a sign) from the N–S axis of rotation $(-1 \leqslant \rho \leqslant 1)$. If ρ and z are considered as Cartesian coordinates in the meridian plane of P, then ρ changes sign when the path passes through a pole. θ is the angle between the rays drawn from the center of the sphere to our position and to the North Pole $(0 \leqslant \theta \leqslant \pi)$.

In order to safeguard the validity of Eq. (2.2) even in the case when the flight path passes through a pole of the Earth, we have to agree on a special rule for ϕ (see Fig. 1).

When passing through a pole, even though s increases only by an infinitesimal amount, according to the conventional rule ϕ would suddenly increase (or decrease) by π, thus violating Eq. (2.2). To avoid this we posit that ϕ remains unchanged when the path passes through the South Pole. This rule is in accordance with the view that when the point P passes through the South Pole it remains in the same meridian plane as before the passage. With this rule adopted, there is no problem at the passage through the North Pole. This rule is also in accord with our conventions, stated ear-

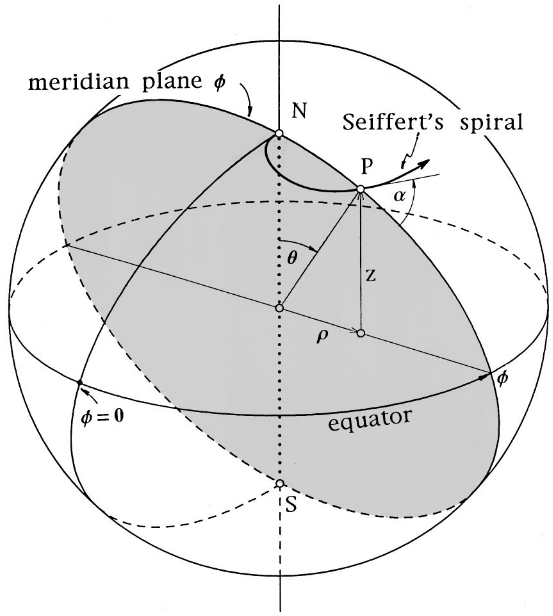


Fig. 1. The beginning of A. Seiffert's spiral on the unit sphere. The tangent to the spiral at the North Pole defines $\phi=0$. The coordinates ϕ, z, ρ, and θ are illustrated, as well as the angle α between the tangent to the curve and the meridian ϕ at an arbitrarily chosen point P on the spiral.

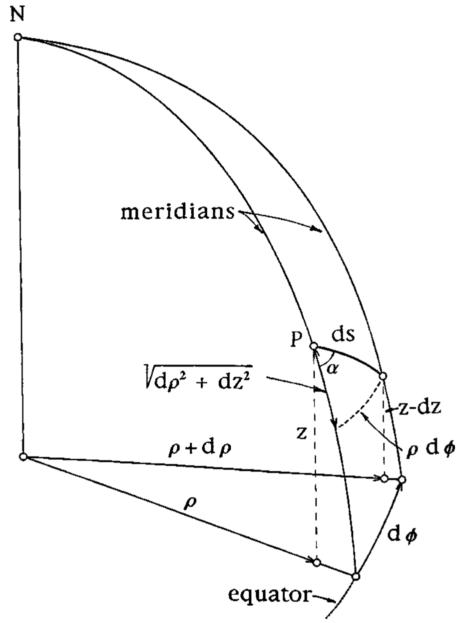


Fig. 2. An infinitesimal line element ds on the surface of the unit sphere. The square of the line element is expressed by a choice of the coordinates ϕ, z, and ρ in Eqs. (2.3) and (2.4). Opposite ds there is a right angle in the infinitesimal triangle shown.

lier, that ρ is positive in that half of the meridian plane where the spiral moves from the North to the South Pole and is negative in the other half.

Using an infinitesimal line element ds on the sphere (see Fig. 2), it is clear that the square of the line element of any curve on the unit sphere may be expressed as

$$ds^2=\rho^2\,d\phi^2+d\rho^2+dz^2=\rho^2\,d\phi^2+\frac{1}{1-\rho^2}\,d\rho^2,\quad \rho\neq 1, \tag{2.3}$$

where we have used $\rho^2+z^2=1$ to eliminate dz^2.

We now make use of the special property, Eq. (2.2), which relates the longitude ϕ to the distance s traveled by the flying machine, to replace $d\phi^2$ by $k^2\,ds^2$ in Eq. (2.3) and obtain

$$ds=\frac{1}{\sqrt{(1-k^2\rho^2)(1-\rho^2)}}\,d\rho\quad\left(|\rho|<\frac{1}{k}\right). \tag{2.4}$$

Hence, the total distance s traveled from the North Pole to some point P may be expressed by the distance ρ of that point from the N–S axis as

$$s(\rho,k)\equiv\int_0^{\rho}\frac{d\rho'}{\sqrt{(1-k^2\rho'^2)(1-\rho'^2)}}\quad\left(|\rho|<\frac{1}{k}\right). \tag{2.5}$$

We recognize $s(\rho,k)$ as the elliptic integral of the first kind with parameter k. Since the square of k occurs in Eq. (2.5), many treatises use

$$m=k^2,$$

a notation which we also adopt to designate the parameter of the elliptic integral.

Instead of expressing s as a function of ρ, as in Eq. (2.5), we may imagine ρ expressed as a function of s. This inverse function is Jacobi's elliptic function $\mathrm{sn}(s|m)$, hence

$$\rho=\mathrm{sn}(s|m). \tag{2.6}$$

We may choose to eliminate $d\rho$ rather than dz from Eq. (2.3). In this fashion we obtain an expression of the length of the path traveled as a function of the distance z of P from the equatorial plane:

$$s(z,k)=\int_1^{z}\frac{dz'}{\sqrt{(1-k^2+k^2z'^2)(1-z'^2)}},\quad \overset{+}{\sqrt{1-z^2}}<\frac{1}{k}. \tag{2.7}$$

The inverse of $s(z,k)$ is Jacobi's elliptic function $\mathrm{cn}(s|m)$:

$$z=\mathrm{cn}(s|m) \tag{2.8}$$

and from $\rho^2+z^2=1$ it follows that

$$\mathrm{sn}^2(s|m)+\mathrm{cn}^2(s|m)=1. \tag{2.9}$$

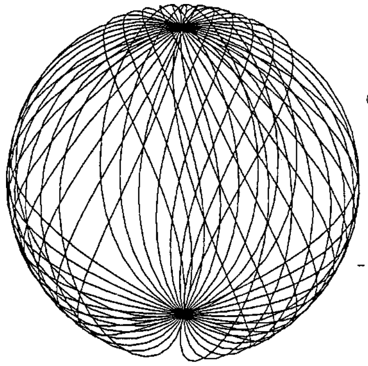
The notation sn and cn, and Eq. (2.9), remind us of sine and cosine: This is not a chance coincidence, as we shall see. First, however, let us study the space curve defined by the following three equations (2.2), (2.6), and (2.8), summarized as:

$$\begin{aligned}&\phi=ks,\quad \text{where}\ \ k=\sqrt{m},\\ &\rho=\mathrm{sn}(s|m),\quad z=\mathrm{cn}(s|m).\end{aligned} \tag{2.10}$$

These equations define Seiffert's spiral, situated on the surface of the unit sphere [see Eq. (2.9)], given in cylindrical (not spherical!) coordinates as functions of the length s of the curve measured from the North Pole ($\phi=0,\rho=0,z=1$) and containing a parameter m.

III. PROPERTIES OF A. SEIFFERT'S SPHERICAL SPIRAL

We recall that all Seiffert spirals correspond to the same angular velocity of flight, but to different surface velocities distinguished by the parameter m. In Fig. 2 we see a schematic example of a piece of the curve traced out by our



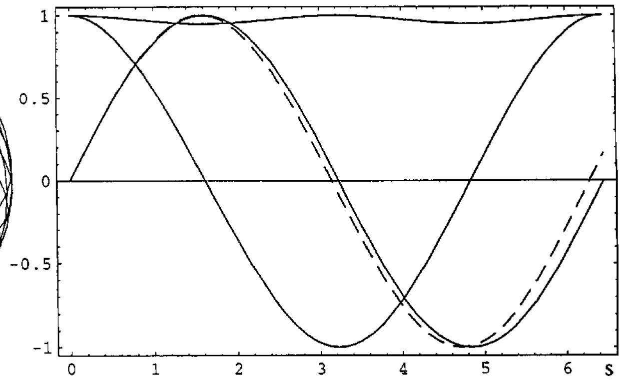


Fig. 3. *Left*:A piece of A. Seiffert's spiral for $m=0.1$, of length $S=50\pi$. *Right*: The three Jacobian elliptic functions $\mathrm{sn}(s|0.1)$ (starting at 0), $\mathrm{cn}(s|0.1)$ (starting at 1), $\mathrm{dn}(s|0.1)$ (staying close to 1), and $\sin(s)$ (broken line). The period of $\mathrm{sn}(s|0.1)$ and $\mathrm{cn}(s|0.1)$ is $4K(0.1)=6.45$; the period of $\mathrm{dn}(s|0.1)$ is $2K(0.1)$. Here, $K(m)$ is the complete elliptic integral of the first kind. For a fixed value of m, the arclength s, measured between the North Pole and any chosen point P, of the spiral defined by $\phi=\sqrt{m}s$ and the coordinates ϕ, ρ, and z of the point P are related by Eq. (2.10). The spirals are shown on the left, of Figs. 3–6. The coordinates ρ and z yield the values of $\mathrm{sn}(s|m)$ and $\mathrm{cn}(s|m)$, respectively, depicted on the right of these figures as functions of s.

flying machine for some parameter m. At the arbitrary point P the curve crosses the meridian at an angle α and from the infinitesimal spherical triangle at P we have

$$\cos\alpha=\frac{\sqrt{dz^2+d\rho^2}}{ds}=\sqrt{1-m\,\mathrm{sn}^2(s|m)}. \tag{3.1}$$

To obtain the last equality we applied the relationships utilized to obtain Eq. (2.4) and the definition of Eq. (2.6).

The right-hand side of Eq. (3.1) may be taken as the defining relationship of another of the Jacobian elliptic functions, denoted by $\mathrm{dn}(s|m)$:

$$\mathrm{dn}(s|m)=\cos\alpha. \tag{3.2}$$

Hence, if we travel a distance s from the North Pole along Seiffert's spiral, the cosine of the angle formed at that point with the N–S meridian equals the Jacobian elliptic function $\mathrm{dn}(s|m)$.

Consider[5] the case $m=0$. The first equation of (2.10) shows that ϕ remains zero for any s. Therefore we travel along the meridian from the North to the South Pole as s increases from 0 to π and by looking at the meridian circle (Fig. 1) we see that $s=\theta$, $\rho=\sin s$, $z=\cos s$, $\alpha=0$. From Eqs. (2.10) and (3.2) it follows that

$$\mathrm{sn}(s|0)=\sin s,\quad \mathrm{cn}(s|0)=\cos s,\quad \mathrm{dn}(s|0)=1. \tag{3.3}$$

Equations (3.3) show that the trigonometric functions are special cases of the Jacobian elliptic functions.

For travelers, it is quite disagreeable to change direction suddenly or what amounts to the same thing, to have cusps in the flight curve. Therefore, at the South Pole we do not reverse direction and mount along the meridian $\phi=0$, but rather we continue smoothly and, according to the rule adopted in Sec. II, ϕ remains zero. Thus we complete a full meridian circle. The rule concerning ϕ will also ensure the absence of cusps for $m>0$. Of course, points differing in their ϕ coordinates by multiples of 2π are to be considered identical.

When $m\neq 0$ but $m\ll 1$, the surface velocity of the traveler is high compared to the velocity he/she would need to have if he/she wanted to fly around the equator in one day ($m=1$). To limit his angular velocity, the N–S component of v has to be large compared to its E–W component. Hence, the curve defined by Eq. (2.10) will make a small angle α with the meridians it crosses and $\mathrm{dn}(s|m)$ will remain close to 1 for all s (Fig. 3). Equation (2.7) shows that $s(z,k)$ exists (i.e., the integral converges) for all $|z|\leq 1$, as long as $m<1$, therefore the spiral will always pass through the South Pole before rising again to the North Pole, and it will continue covering the sphere as shown in Fig. 3, passing alternately through its poles. Figure 3 shows that the spiral is antisymmetric with respect to a reflection on the equatorial plane (the proof is left to the reader), that it is periodic in a sense that will be discussed in Sec. V, and that the elliptic functions sn and cn for $m=0.1$ are still close to their trigonometric counterparts.

Apart from certain special cases (see Sec. IV), every point of the sphere is crossed twice by the spiral, except the North and South Poles, which are crossed infinitely many times. The length of the spiral is infinite, unless it forms a closed curve (see Sec. VI).

Figure 4 shows an example for $m=0.8$. Here, the Jacobian elliptic functions already differ markedly from the trigonometric functions and, since the spiral section between two successive pole crossings is considerably longer than the length π of half a meridian circle, the half-period (see Sec. V) is $2K(0.8)\cong 4.51$.

As we let m increase toward 1, the angle α_0 at which the spiral crosses the meridian at the equator approaches 90°. This follows from the fact that $\rho=\mathrm{sn}(s|m)=1$ on the equator, and Eq. (3.1) with $m=1$ then implies $\mathrm{dn}(s|m)=\cos\alpha=0$. This means that for $m=1$ the spiral never crosses the equator. In fact, it winds around the globe in the northern hemisphere, asymptotically approaching the equator, because its length s, given by the integral in Eq. (2.7), becomes infinite for $z=0$ in the case $m=1$. The three elliptic functions and a finite portion of Seiffert's spiral are shown in Fig. 5 for this exceptional case $m=1$.

For $m=k=1$, Eq. (2.2) reduces to $\phi=s$ and from Fig. 2 we see that

$$\sin\alpha=\frac{\rho\,d\phi}{ds}=\rho. \tag{3.4}$$

On the other hand, since $\sin\theta=\rho$, it follows that

$$\alpha=\theta. \tag{3.5}$$

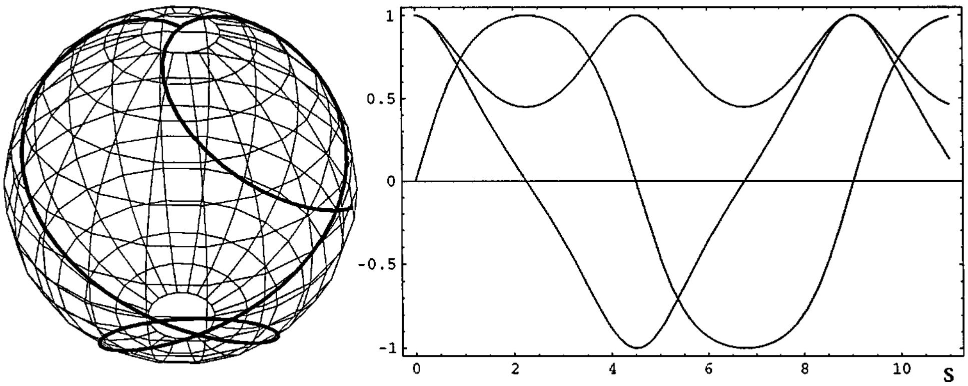


Fig. 4. *Left*: Heavy line: The section between $s=0$ and $s=3.5\pi$ of A. Seiffert's spiral for $m=0.8$. The parameter s is the arclength. The full period of the spiral (return to the North Pole) is $4K(0.8)=2.87\pi$. The skeleton outline of the unit sphere is shown for guidance. *Right*: The functions $\mathrm{sn}(s|0.8)$ (starting at 0), $\mathrm{cn}(s|0.8)$ (starting at 1), $\mathrm{dn}(s|0.8)$ (always >0) for $0\leqslant s\leqslant 3.5\pi$.

Hence, for $m=1$, the spiral has the property that at every point the angle α that it makes with the meridian equals the latitude θ. Moreover, the integrals in Eqs. (2.5), (2.7), and (5.1) can be expressed by inverse hyperbolic functions and it follows that

$$\mathrm{sn}(s|1)=\tanh(s),$$
$$\mathrm{cn}(s|1)=\mathrm{dn}(s|1)=\mathrm{am}(s|1)=\mathrm{sech}(s). \tag{3.6}$$

(The function $\mathrm{am}(s/m)$ will be defined in Sec. V.)

IV. LOW SPEED TRAVEL

Up to this point we have considered spirals with $m\leqslant 1$. Now we want to investigate Seiffert's spiral for $m>1$. This case arises when the surface speed of the flying machine is low. If the angular velocity is to remain the same as at high surface speed, the flying machine must stay close to the North Pole, so as to circle the globe in a short time. Indeed, when $m>1$ the spiral stays entirely in the northern hemisphere. Without any calculations we may conclude that when $m\gg 1$, ρ remains small for all s, hence the function $\mathrm{sn}(s|m)$ will be small in absolute value. In contrast, z, and therefore $\mathrm{cn}(s|m)$, will be close to 1 for all s in this case. Figure 6 shows the three elliptic functions for $m=10$. We denote the lowest point of the spiral by the subscript L. At L the spiral is tangent to a circle of latitude, hence $\cos\alpha_L=0$. From Eq. (3.1) it follows that at that point

$$\rho_L=\max[\mathrm{sn}(s|m)]=\frac{1}{\sqrt{m}} \quad \text{for } m>1. \tag{4.1}$$

ρ_L is the maximum distance of the spiral from the N–S axis. From Eq. (2.9),

$$z_L=\min[\mathrm{cn}(s|m)]=\sqrt{1-\frac{1}{m}} \quad \text{for } m>1: \tag{4.2}$$

z_L is the latitude of the lowest point of the spiral.

Figure 7 shows the spiral and the three elliptic functions for $m=1.25$.

If one were only interested in the elliptic functions themselves and not in Seiffert's spiral, it would not be indispensable to study the case $m>1$, because there exist certain reciprocity relations which allow one to express the elliptic functions for any $m>0$ in terms of those for $1/m$. We will derive these relations, again by geometric considerations in Sec. VII.

V. THE PERIODICITY OF THE JACOBIAN ELLIPTIC FUNCTIONS

We limit our discussion to the Jacobian elliptic functions of real arguments s. For $m\leqslant 1$, Seiffert's spiral is close to the meridian circle and its segments lying between any two successive passages of the same pole are congruent in that they can be mapped into each other, together with their direction of passage, by a rotation around the N–S axis. We leave the proof of this congruence to the reader. Because of this congruence, a segment between two successive passages of the same pole is considered the *period* of the spiral. The length S of the arc between the pole and the equator, which of course depends on m, is called the complete elliptic integral $K(m)$

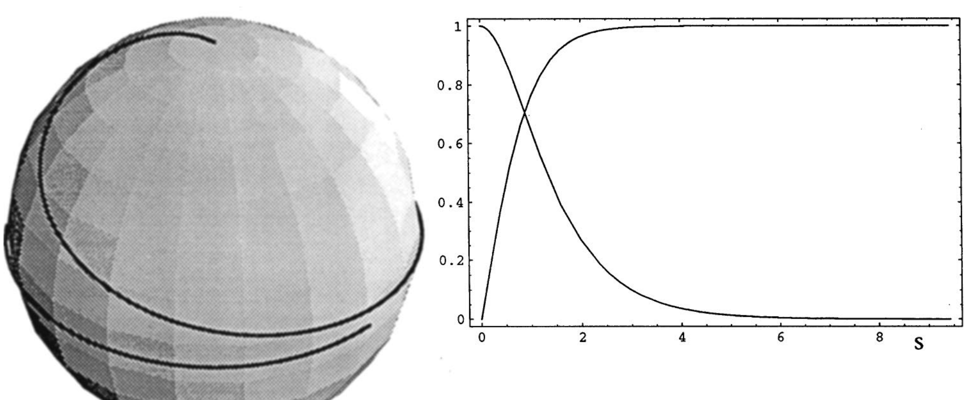


Fig. 5. *Left*: A. Seiffert's spiral for $m=1$ shown on the unit sphere (simulated by the shaded surface) between the arclength parameter values $s=0$ and $s=3\pi$. The curve approaches the equator asymptotically. *Right*: The three Jacobian elliptic functions $\mathrm{sn}(s|1)\equiv\tanh(s)$ (starting at 0) and $\mathrm{cn}(s|1)=\mathrm{dn}(s|1)\equiv\mathrm{sech}(s)$ between $s=0$ and $s=3\pi$.

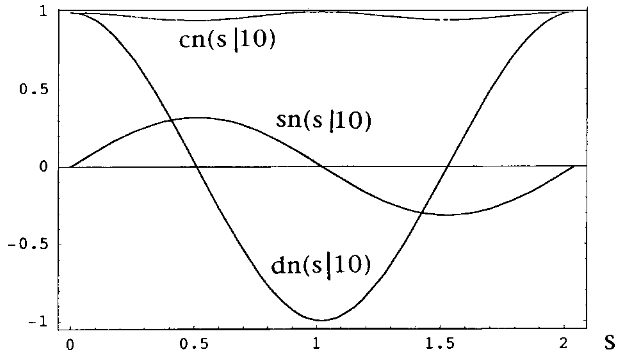


Fig. 6. The Jacobian elliptic functions for $m=10$ in the interval between $s=0$ and $s=4K(10)=2.04$. From Eqs. (4.1) and (4.2), $\rho_L=0.31$, $z_L=0.949$.

of the first kind, and is given by Eq. (2.5) with the upper limit $\rho=1$. The total length of the period is thus $4S$, i.e., $4K(m)$, and this is also the period of $\rho=\mathrm{sn}(s|m)$ and $z=\mathrm{cn}(s|m)$, which return to their starting values for the first time with the same derivative after a full period of the spiral.

Unlike the other two Jacobian elliptic functions, the period of $\mathrm{dn}(s|m)$ is $2K(m)$. This can be understood from the fact that dn, which is the cosine of the angle between the spiral and the meridian, equals unity at each pole crossing; hence, the period of $\mathrm{dn}(s|m)$ is $2S$.

We take this opportunity to introduce here one more (for this paper, the last) elliptic function: the so-called Jacobian amplitude $\mathrm{am}(s|m)$. It turns out that this is simply the angle θ introduced with reference to Fig. 1. In fact, the usual definition of $\mathrm{am}(s|m)$ is that of being the inverse function of the integral

$$s(\theta,k)=\int_0^\theta \frac{d\theta'}{\sqrt{1-k^2\sin^2\theta'}}. \tag{5.1}$$

Expressing ds^2 in terms of θ and $d\theta$, using $\rho=\sin\theta$, $z=\cos\theta$, and $d\phi^2=k^2\,ds^2$, the relation Eq. (5.1) follows immediately. Therefore, θ is the inverse function of $s(\theta,m)$ and

$$\mathrm{am}(s|m)=\theta(s|m). \tag{5.2}$$

It follows from the geometry of the sphere that

$$\mathrm{sn}(s|m)=\sin \mathrm{am}(s|m), \tag{5.3}$$

$$\mathrm{cn}(s|m)=\cos \mathrm{am}(s|m), \tag{5.4}$$

and, as easily verified,

$$m\,\mathrm{sn}^2(s|m)+\mathrm{dn}^2(s|m)=1. \tag{5.5}$$

Since θ takes the same value if we add $4S$ to the length of the spiral, the period of $\mathrm{am}(s|m)$ is $4K(m)$.

As was shown in Sec. IV, when $m>1$ Seiffert's spiral stays in the northern hemisphere, and this fact changes some of the periodicity properties of the Jacobian elliptic functions. The coordinate ρ still has the same period $4K(m)$ as for $m<1$, since it returns to zero with the same slope $d\rho/ds$ at every second crossing of the North Pole (see Fig. 7), but z and $\cos\alpha$ reverse their roles. For details, see Eq. (7.5) of this paper.

VI. SEIFFERT'S SPHERICAL SPIRAL AS A CLOSED CURVE

We would like to find out under what conditions Seiffert's spiral is a periodic function on the sphere, or in other words, when does it form a closed curve? This question is different from the question concerning the periodicity of the elliptic functions since these are always periodic except when $m=1$. Precisely in this case, the spiral cannot be a closed curve, since it approaches the equator in an asymptotic fashion. As we shall see, when $m\neq 1$ there are infinitely many values of m for which the spirals form closed curves with interesting properties.

We first consider the case $m<1$, for which the spiral always passes alternately through the North and South Poles of the unit sphere. The condition for the spiral to be a closed curve may be expressed by requiring that the flying machine, starting at the North Pole at some longitude ϕ_0 and having traveled some distance s_n, returns to the North Pole at the angle

$$\phi_n=\phi_0+2\pi n \quad \text{with } n \text{ a positive integer.} \tag{6.1}$$

This passage may not be the first return to the North Pole, but we assume that it is the first return at the angle ϕ_n. In this situation, the returning branch of the spiral coalesces smoothly with the starting branch and the curve is periodic. Without loss of generality we may set $\phi_0=0$. Since at the North Pole $\rho=0$, we require [see Eq. (2.10)]

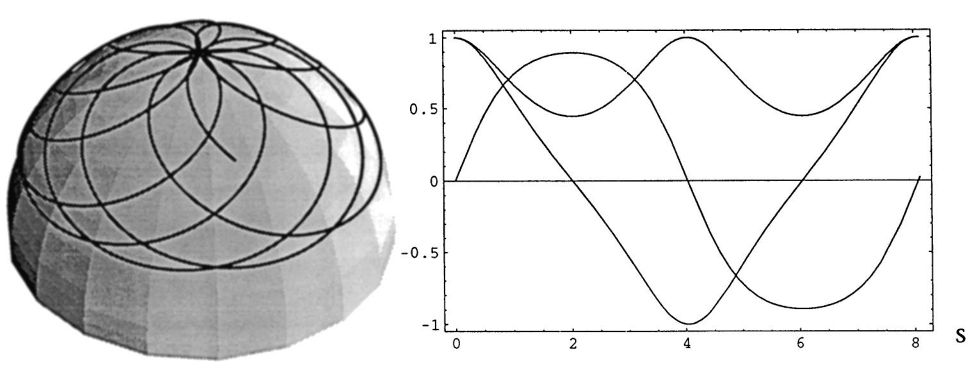


Fig. 7. *Left*: The spiral for $m=1.25$. For $m>1$, the curve stays in the northern hemisphere above the latitude $\theta_L=\arcsin(m^{-1/2})=63°$. A piece of length $s=32$ is shown. *Right*: The three Jacobian elliptic functions for $m=1.25$. Because of the reciprocity relations (see Sec. VII), $\mathrm{sn}(s|1.25)$ (starting at 0) can be scaled to coincide with $\mathrm{sn}(s|0.8)$ and $\mathrm{cn}(s|1.25)$ (always positive) and $\mathrm{dn}(s|1.25)$ can be scaled to coincide, respectively, with $\mathrm{dn}(s|0.8)$ and $\mathrm{cn}(s|0.8)$ [compare with Fig. 4; note that, according to Eqs. (7.6)–(7.8), both the ordinate and abscissa have to be scaled].

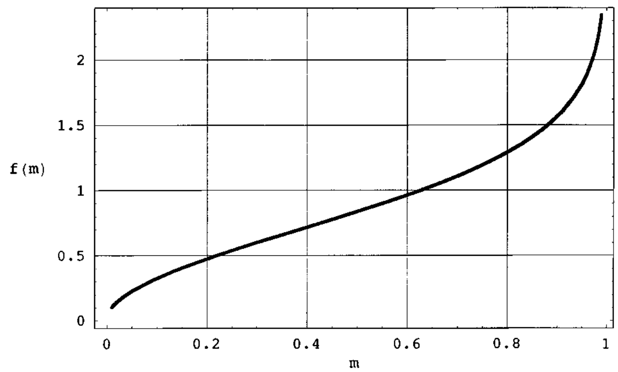


Fig. 8. The function $f(m)=2\sqrt{m}K(m)/\pi$ plotted against m, the parameter of the Jacobian elliptic functions: $K(m)$ is the complete elliptic integral of the first kind. For those values of m for which $f(m)$ is a rational number, Seiffert's spiral is a closed curve.

$$\mathrm{sn}(s_n|m)=0. \tag{6.2}$$

Using $s=\phi/\sqrt{m}$, $(m\neq 0)$ [Eq. (2.10)], the condition of return at the angle ϕ_n is[6]

$$s_n=\frac{2\pi}{\sqrt{m}}n, \quad \text{with } n \text{ a positive integer.} \tag{6.3}$$

Here, s_n denotes the length of the closed spiral.

Combining Eqs. (6.1) and (6.3), this leads to

$$\mathrm{sn}\left(\frac{2\pi}{\sqrt{m}}n|m\right)=0. \tag{6.4}$$

Since $\mathrm{sn}(s|m)$ is a periodic function with period $4K(m)$, where $K(m)$ is the complete elliptic integral of the first kind, and $\mathrm{sn}(0|m)=0$, Eq. (6.4) requires

$$\frac{2\pi}{\sqrt{m}}n=4K(m)p, \quad p=\text{integer}.$$

Hence, the spiral will be a closed curve if m is so chosen that

$$f(m)\equiv\frac{2}{\pi}\sqrt{m}K(m)=\frac{n}{p}, \quad \text{with } p \text{ and } n \text{ positive integers,} \quad m<1. \tag{6.5}$$

We suppose that common factors of p and n have been canceled out, since they are irrelevant to our problem.

The function $f(m)$ defined by Eq. (6.5) varies between 0 and ∞ when m varies between 0 and 1 (see Fig. 8). The rational numbers n/p form a dense set over every interval from 0 to ∞; therefore there will be infinitely many closed spirals for any interval of values of m. Of course, the value of m which corresponds to a particular n/p can only be found by a numerical, and therefore approximate, solution of the transcendental equation (6.5).

The integers p and n each have an interesting geometrical interpretation. Since p is the number of periods of $\mathrm{sn}(s|m)$ completed before the spiral closes for the first time and in every period $\rho=\mathrm{sn}(s|m)$ has two zeros, p equals the number of times the closed curve passes each pole. On the other hand, n gives the number of times the curve circles the N–S axis. Some examples of these closed spirals are shown in Fig. 9.

When $m>1$, the spiral is located entirely in the northern hemisphere. Since it does not reach the equator, the complete elliptic integral does not play a role in the question of periodicity, as it does for $m<1$. Without going into details, we state that periodicity occurs for those values of m for which the following equation holds:

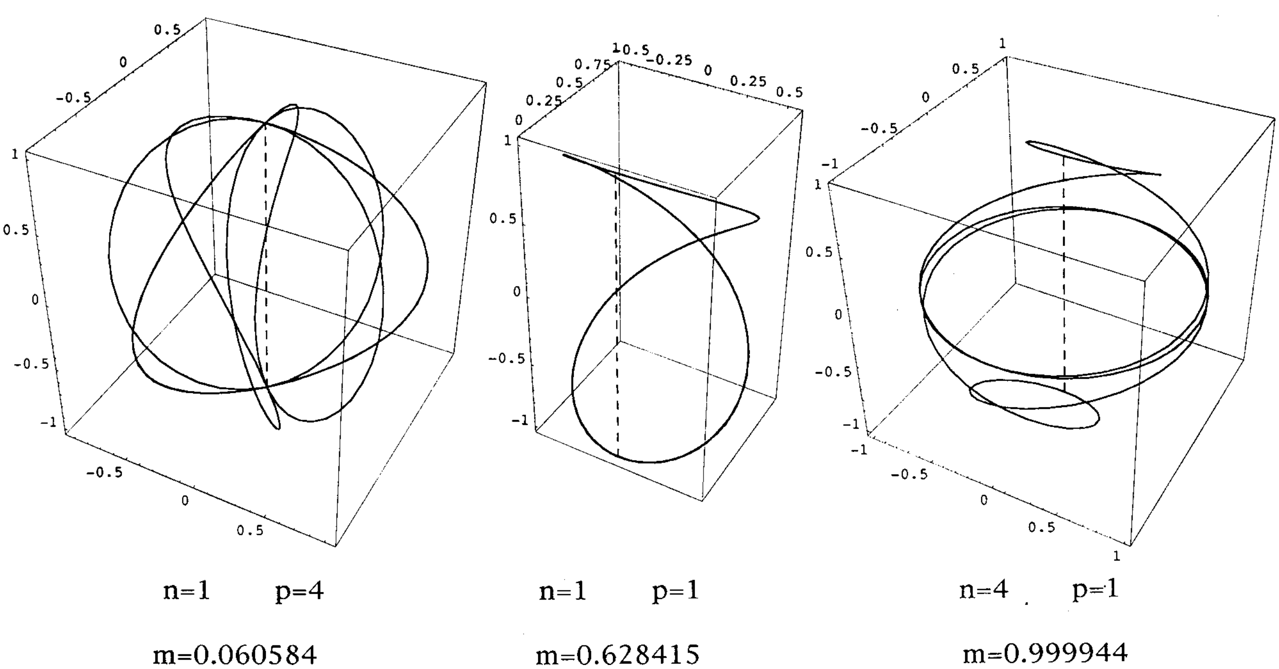


Fig. 9. Examples of Seiffert's closed spirals for a choice of parameter values $m<1$ which satisfy Eq. (6.5), i.e., $f(m)=n/p$. The spiral passes each pole p times and circles the sphere n times.

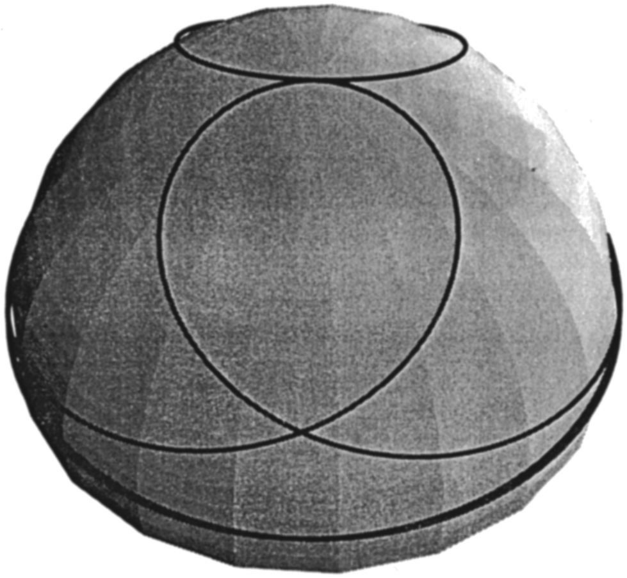


Fig. 10. An example of a closed Seiffert's spiral for $m>1$, where $m=1.000\,055\,809\,6$ is a solution of Eq. (6.6) with $p=4$ and $n=1$.

$$\mathrm{dn}\left(\frac{\pi}{2\sqrt{m}}\frac{p}{n}|m\right)=0,\quad \text{with } p \text{ and } n \text{ positive integers, } m>1. \tag{6.6}$$

Figure 10 shows an example of the closed spiral for $m>1$.

VII. RECIPROCITY RELATIONS

The Jacobian elliptic functions obey certain reciprocity relations [7] which connect the functions with parameter m to functions with parameter $1/m$.

It is thus sufficient to study the behavior of these functions for $m<1$. Nevertheless, we have discussed Seiffert's spirals for all $m>0$ because the analytic relationships between the functions with reciprocal parameters cannot be translated into simple geometric relationships between the corresponding spirals.

Despite this fact, Sieffert's construction can be used to derive the reciprocity relations, as we shall now show.

Let us consider one of Seiffert's spirals, sp 1, constructed for $m<1$ on the unit sphere, and imagine a second concentric sphere of radius

$$R=\frac{1}{\sqrt{m}} \tag{7.1}$$

being placed around the unit sphere. We project sp 1 upon the northern half of the larger sphere, the projection being *parallel* to the N–S axis. We call the curve so obtained sp 2. To anticipate, let us remark that even though sp 2 is a spiral, it is not a Seiffert spiral insofar as there is no linear proportionality [8] between its arclength s' and the meridian angle ϕ', i.e., $\phi'\neq(\text{const})s'$. This means that if we wish to fly at constant angular velocity $d\phi/dt$ along sp 2, we have to vary our velocity ds'/dt. Nevertheless, sp 2 has some interesting properties. If we denote the coordinates on the larger sphere by ϕ', ρ', θ', z' by analogy with the unprimed coordinates on the unit sphere, it is clear from the projection method used to create sp 2 that

$$\phi'=\phi,\quad \rho'=\rho=\sin\theta=R\sin\theta'. \tag{7.2}$$

From Eq. (2.4) we eliminate ρ in favor of θ' using Eqs. (7.2) and (7.1) to obtain

$$\sqrt{m}\,ds=\frac{d\theta'}{\sqrt{1-\frac{1}{m}\sin^2\theta'}}. \tag{7.3}$$

Since the integral of the right-hand side of Eq. (7.3) is the elliptic integral of the first kind in its standard form, its inverse function is the Jacobian elliptic function; hence,

$$\sin\theta'=\mathrm{sn}\left(\sqrt{m}s\,\middle|\,\frac{1}{m}\right). \tag{7.4}$$

On the other hand, from Eq. (2.10)

$$\sin\theta=\mathrm{sn}(s|m), \tag{7.5}$$

therefore, using Eq. (7.2), it follows that

$$\mathrm{sn}\left(\sqrt{m}s\,\middle|\,\frac{1}{m}\right)=\sqrt{m}\,\mathrm{sn}(s|m). \tag{7.6}$$

This is the desired relationship connecting the sn functions with reciprocal parameters. Finally, with Eqs. (2.9) and (3.1) we deduce from Eq. (7.6) that

$$\mathrm{cn}\left(\sqrt{m}s\,\middle|\,\frac{1}{m}\right)=\mathrm{dn}(s|m) \tag{7.7}$$

and

$$\mathrm{dn}\left(\sqrt{m}s\,\middle|\,\frac{1}{m}\right)=\mathrm{cn}(s|m). \tag{7.8}$$

It should be remembered that s is the length of arc of the original, not the projected, spiral.

Figure 11 shows Seiffert's spiral for $m=0.358$, together with its northward parallel projection onto the sphere of radius $R=1/\sqrt{m}$. The spheres are not shown.

VIII. STARTING THE TRIP ANYWHERE

The North Pole may not be everybody's favorite starting point for circling the globe in one day. We have chosen this starting point to facilitate the comparison of spirals of different parameters m. Given a certain surface velocity v, if $m=a^2/v^2<1$ ($a=2\pi$ per day, see Sec. II), the starting point of the spiral can be anywhere on the sphere.

If the velocity is small, so that $m>1$, then the coordinate z_P of the starting point has to satisfy the condition, arising from Eq. (4.2), that

$$z_P>z_L=\sqrt{1-\frac{1}{m}}, \tag{8.1}$$

because in this case the spiral stays on the part of the sphere where $z>z_L$.

In either case, the spiral obtained starting at P is just a part of the Seiffert's spiral with parameter m which passes through P and which we have studied in the previous sections.

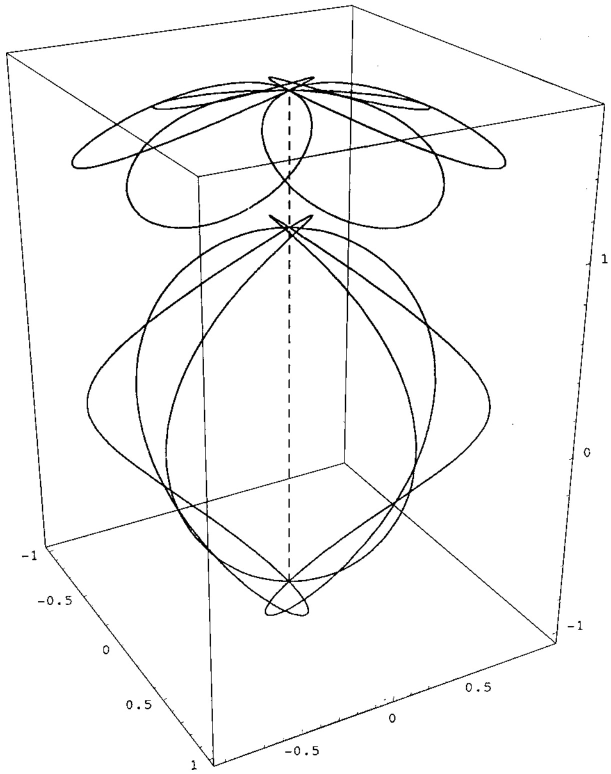


Fig. 11. An example of a Seiffert spiral with $m=0.358$ and its parallel projection on a concentric sphere of radius $R=1/\sqrt{m}$. The properties of the projected curve are used to derive the reciprocity relations, Eqs. (7.6)–(7.8) of the Jacobian elliptic functions. Since Eq. (6.5) is satisfied with $n=2$ and $p=3$, the curves are closed.

Our last remark concerns those travelers who wish to circle the Earth during a time period other than one day. Inspired by literature, they may choose eighty days, or if aspiring to set a record, they may want to do the trip in less than a day. It is left to the readers to convince themselves that, as long as the conditions of constant surface and angular velocities are maintained, they must travel on a Seiffert's spiral.

ACKNOWLEDGMENTS

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[1]C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum* (Königsberg, 1829).

[2]Alfred Seiffert, ''Über eine neue geometrische Einführung in die Theorie der elliptischen Funktionen,'' Wissenschaftliche Beilage zum Jahresbericht der Städtischen Realschule zu Charlottenburg, Ostern, 1896.

[3]E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge U.P., Cambridge, 1958), pp. 527–528.

[4]Instead of using the geographic East and West longitude, we measure ϕ increasing in the direction opposite to the Earth's rotation, $\phi \geqslant 0$, and keep calling this quantity ''longitude.''

[5]For this special case one would have to set $v=\infty$, but Eq. (2.10) remains valid.

[6]The case $n=0$ represents the trivially closed meridian circle.

[7]L. M. Milne-Thomson, ''Jacobian Elliptic Functions and Theta Functions,'' in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972), 9th printing, p. 573.

[8]One can show that the lengths s' of sp 2 and s of sp 1 are related by $s' = \int_0^s \sqrt{1-(1-m)\mathrm{sn}^2(x|m)}\,dx$, and since s is linearly proportional to ϕ, s' is not.

SPEAKING HUNGARIAN

Fermi, Rossi, I, and perhaps some other Italian-speaking physicist, were lunching one day during this period at Fuller Lodge, and as usual, we slipped into Dante's language; as usual, talking loudly. General Groves was nearby, and he let us know that he did not like us speaking Hungarian (!) in public; he delicately hinted that if we wanted to speak foreign languages, we had better go into the woods.

Emilio Segrè, *A Mind Always in Motion—The Autobiography of Emilio Segrè* (University of California Press, Berkeley, 1993), p. 182.