\[ f(z) = \frac{2}{z^2 + 1} = \frac{(x + iy)^2}{(x^2 - y^2 + 1 + 2xyi)(x^2 - y^2 + 1 - 2xyi)} \]

\[ = \frac{x^2 + xy^2 + x + i(y - x^2 - y^2)i}{(x^2 - y^2 + 1)^2 + 4x^2y^2} \]

\[ \frac{\partial u}{\partial x} = \frac{[(x^2 - y^2 + 1)^2 + 4x^2y^2][3x^2 + y^2 + 1] - [(x^2 + xy + x)(2x - y^2 + 1)2x + 8xy]^2}{[(x^2 - y^2 + 1)^2 + 4x^2y^2]^2} \]

\[ \frac{\partial u}{\partial x} = \frac{[(x^2 - y^2 + 1)^2 + 4x^2y^2][1 - x^2 - 3y^2 - (y - x^2 - y^2)]2x^2 + 8x^2y}{[(x^2 - y^2 + 1)^2 + 4x^2y^2]^2} \]

Expanding numerator \( \frac{\partial u}{\partial x} \)

\[ = -x^6 + 4x^4y^2 + 4x^2 - y^2 + 10x^2y^2 - x^2 - y^2 + x^3y + ry^2 - 1 \]

Expanding numerator \( \frac{\partial v}{\partial y} \)

\[ = -x^6 + x^4y^2 + 4x^2 - y^2 + 10x^2y^2 - x^2 - y^2 + y^3 + y^3 - 1 \]

\[ \frac{\partial u}{\partial y} = \frac{[(x^2 - y^2 + 1)^2 + 4x^2y^2](2xy) - [(x^2 + xy + x)(2x + 8xy)]}{[(x^2 - y^2 + 1)^2 + 4x^2y^2]^2} \]

\[ \frac{\partial v}{\partial x} = \frac{[(x^2 - y^2 + 1)^2 + 4x^2y^2][-2xy] - [(y - x^2 - y^2)]2x + 8x^2y}{[(x^2 - y^2 + 1)^2 + 4x^2y^2]^2} \]

Expanding numerator \( \frac{\partial u}{\partial y} \)

\[ = -2xy(x^4 + 2x^2y^2 - 2x^2y^2 + y^4 + y^4 + y^4 - 3) \]

Expanding numerator \( \frac{\partial v}{\partial x} \)

\[ = 2xy(x^4 + 2x^2y^2 - 2x^2y^2 + y^4 + y^4 + y^4 - 3) \]

Since \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \), \( f(z) \) is analytic except at \( z = \pm i \).

\[ f(z) = e^z = e^{x(\cos(y) + i\sin(y))} = e^x \cos(y) - ie^x \sin(y) \]

\[ \frac{\partial u}{\partial x} = e^x \cos(y), \quad \frac{\partial v}{\partial y} = -e^x \cos(y), \quad \frac{\partial u}{\partial y} = -e^x \sin(y), \quad \frac{\partial v}{\partial x} = -e^x \sin(y) \]

Since \( u_x = \neq v_y \) and \( v_x = -u_y \), \( f(z) \) is not analytic.

\[ f(z) = \frac{y - \frac{x}{x^2 + y^2}}{x^2 + y^2}, \quad u_x = -\frac{2xy}{(x^2 + y^2)^2}, \quad v_y = \frac{2xy}{(x^2 + y^2)^2}, \quad u_y = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad v_x = \frac{x^2 - y^2}{(x^2 + y^2)^2} \]

Since \( u_x \neq v_y \) and \( v_x \neq -u_y \), \( f(z) \) is not analytic.
14.3.6  

\[ a. \int_{C} z \, dz \quad \text{along } C \in \mathbb{C} \quad z = e^{i \theta} \quad \theta \leq \theta \leq \frac{\pi}{2} \]

\[ \oint_{C} z \, dz = \int_{0}^{\frac{\pi}{2}} e^{i \theta} \, i \cdot e^{i \theta} \, d\theta = i \int_{0}^{\frac{\pi}{2}} e^{2i \theta} \, d\theta = i \left. \frac{e^{2i \theta}}{2i} \right|_{0}^{\frac{\pi}{2}} = \frac{1}{2} (-1 - 1) = -1 \]

\[ b. \text{along } C \]

\[ \oint_{C} z \, dz = \int_{0}^{1} (1 + iy) \, i \, dy + \int_{1}^{0} (x + i \cdot 0) \, dx = (iy - \frac{y^2}{2})|_{0}^{1} + (x - \frac{x^2}{2})|_{1}^{0} = i - \frac{1}{2} - \left( \frac{1}{2} + i \right) = -1 \]

14.7.1. Evaluate:

\[ \int_{0}^{2\pi} \frac{1}{13 + 5 \sin \theta} \, d\theta. \]

Let \( z = e^{i \theta} \). Then we have

\[ dz = ie^{i \theta} \, d\theta \implies d\theta = \frac{1}{iz} \, dz. \]

By definition of \( \sin \theta \), we have

\[ \sin \theta = \frac{e^{i \theta} - e^{-i \theta}}{2i} = \frac{z - 1/z}{2i}. \]

Let \( I \) be the integral

\[ I = \int_{0}^{2\pi} \frac{1}{13 + 5 \sin \theta} \, d\theta. \]

As \( \theta \) runs through \([0, 2\pi]\), \( z = e^{i \theta} \) traces the unit circle \( C \) in the counterclockwise direction. Thus we can make substitutions and turn \( I \) into a contour integral:

\[ I = \oint_{C} \frac{1}{13 + \frac{5}{2i}(z - 1/z)} \cdot \frac{1}{iz} \, dz \]

\[ = 2 \oint_{C} \frac{1}{5z^2 + 26iz - 5} \, dz \]

\[ = 2 \oint_{C} \frac{1}{(z + 5i)(5z + i)} \, dz. \]

This integrand has singularities at \( z = -5i \) and \( z = -i/5 \); only \( z = -i/5 \) lies inside the unit circle \( C \). Thus, we must find the residue of the integrand at \( z = -i/5 \):

\[ R\left(-\frac{i}{5}\right) = \lim_{z \to -i/5} \frac{1}{(z + 5i)(5z + i)} \cdot \frac{1}{(z + 5i)} = \lim_{z \to -i/5} \frac{1}{5(z + 5i)} = \frac{1}{25i - i} = -\frac{i}{24}. \]

By the Residue Theorem,

\[ \oint_{C} \frac{1}{(z + 5i)(5z + i)} = 2\pi i \cdot R\left(-\frac{i}{5}\right) = \frac{\pi}{12}. \]

Thus,

\[ I = 2 \oint_{C} \frac{1}{(z + 5i)(5z + i)} = \frac{\pi}{6}, \]

so

\[ \int_{0}^{2\pi} \frac{1}{13 + 5 \sin \theta} \, d\theta = \frac{\pi}{6}. \]

Wolfram Alpha verifies this result.
Evaluate \[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx. \]

For \( r > 0 \), let \[ S_r = \{ re^{i \theta} \in \mathbb{C} \mid \theta \in [0, \pi) \} \]
be the semicircle of radius \( r \) in the first and second quadrants of the plane and \[ I_r = \{ z \in \mathbb{C} \mid \text{Im}(z) = 0, |z| \leq r \} \]
be the real interval \([-r, r]\) embedded in the complex plane. Then \( C_r = S_r \cup I_r \) is a simple loop in \( \mathbb{C} \). Let \( f : \mathbb{C} \to \mathbb{C} \) be the function \[ f(z) = \frac{1}{z^2 + 4z + 5}. \]

Then \[ \oint_{C_r} f(z) \, dz = \int_{I_r} f(z) \, dz + \int_{S_r} f(z) \, dz \]
\[ = \int_{-r}^{r} f(z) \, dz + \int_{0}^{\pi} f(re^{i \theta}) ire^{i \theta} \, d\theta \]
\[ = \int_{-r}^{r} \frac{1}{x^2 + 4x + 5} \, dx + \int_{0}^{\pi} \frac{ire^{i \theta}}{r^2 e^{2i \theta} + 4re^{i \theta} + 5} \, d\theta. \]

As \( r \to \infty \) we have
\[ \lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ \int_{-r}^{r} \frac{1}{x^2 + 4x + 5} \, dx \right] + \lim_{r \to \infty} \left[ \int_{0}^{\pi} \frac{ire^{i \theta}}{r^2 e^{2i \theta} + 4re^{i \theta} + 5} \, d\theta \right] \]
\[ = \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx + \int_{0}^{\pi} \left[ \lim_{r \to \infty} \frac{ire^{i \theta}}{r^2 e^{2i \theta} + 4re^{i \theta} + 5} \right] \, d\theta \]
\[ = \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx + \int_{0}^{\pi} 0 \, d\theta = \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx. \]

By the Residue Theorem, we also have
\[ \lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ 2\pi i \sum_{i} R(z_i) \right] \]
where \( z_i \) is a singularity of \( f(z) \) inside \( C_r \). The singularities of \( f \) are the roots of the polynomial \( x^2 + 4x + 5 \), which are \( z_{\pm} = -2 \pm i \). The singularity \( z_{-} = -2 - i \) is not inside the curve \( C_r \) for any \( r > 0 \), but for \( r \) large enough, \( z_{+} = -2 + i \) is inside \( C_r \). Thus we have
\[ \lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = 2\pi i R(-2 + i). \]

That is,
\[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx = 2\pi i R(-2 + i). \]

Now we calculate the residue at \( z_{+} = -2 + i \):
\[ R(-2 + i) = \lim_{z \to -2 + i} \left[ (z + 2 - i) \frac{1}{z^2 + 4z + 5} \right] \]
\[ = \lim_{z \to -2 + i} \left[ \frac{(z + 2 - i)}{(z + 2 + i)(z + 2 - i)} \right] \]
\[ = \lim_{z \to -2 + i} \left[ \frac{1}{z + 2 + i} \right] = \frac{1}{2i} = -\frac{i}{2}. \]

Thus,
\[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx = 2\pi i R(-2 + i) = 2\pi i \left(-\frac{i}{2}\right) = \pi. \]

So,
\[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \, dx = \pi. \]

Wolfram Alpha agrees.
14.7.12. Evaluate

\[ \int_{0}^{\infty} \frac{x^2}{x^4 + 16} \, dx. \]

Let \( f : \mathbb{R} \to \mathbb{R} \) be the function given by \( f(x) = x^2/(x^4 + 16) \). Then \( f \) is an even function, so that for \( a > 0 \) we have

\[ \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx. \]

Now consider the extension of \( f \) into the complex plane. For some \( r > 0 \), let

\[ S_r = \{ re^{i\theta} \in \mathbb{C} \mid \theta \in [0, \pi) \} \]

be the semicircle of radius \( r \) in the first and second quadrants of the plane and

\[ I_r = \{ z \in \mathbb{C} \mid \text{Im}(z) = 0, |z| = r \} \]

be the real interval \([-r, r]\) embedded in the complex plane. Then \( C_r = S_r \cup I_r \) is a simple loop in \( \mathbb{C} \). Then

\[ \oint_{C_r} f(z) \, dz = \int_{I_r} f(z) \, dz + \int_{S_r} f(z) \, dz \]

\[ = \int_{-r}^{r} f(x) \, dx + \int_{0}^{\pi} f(re^{i\theta}) ire^{i\theta} \, d\theta \]

\[ = 2 \int_{0}^{r} \frac{x^2}{x^4 + 16} \, dx + \int_{0}^{\pi} \frac{i r^3 e^{3i\theta}}{r^4 e^{4i\theta} + 16} \, d\theta. \]

As \( r \to \infty \) we have

\[ \lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ 2 \int_{0}^{r} \frac{x^2}{x^4 + 16} \, dx \right] + \lim_{r \to \infty} \left[ \int_{0}^{\pi} \frac{i r^3 e^{3i\theta}}{r^4 e^{4i\theta} + 16} \, d\theta \right] \]

\[ = 2 \int_{0}^{\infty} \frac{x^2}{x^4 + 16} \, dx + \int_{0}^{\pi} \left[ \lim_{r \to \infty} \frac{i r^3 e^{3i\theta}}{r^4 e^{4i\theta} + 16} \right] \, d\theta \]

\[ = 2 \int_{0}^{\infty} \frac{x^2}{x^4 + 16} \, dx + \int_{0}^{\pi} 0 \, d\theta = 2 \int_{0}^{\infty} \frac{x^2}{x^4 + 16} \, dx. \]

By the Residue Theorem, we also have

\[ \lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ 2\pi i \sum \text{Res}(z_i) \right] \]

where \( z_i \) is a singularity of \( f(z) \) inside \( C_r \). The singularities of \( f \) are the roots of the polynomial \( x^4 + 16 \), which are \( z_{\pm} = \pm \sqrt{2} \pm i\sqrt{2} \). The singularities \( z_{\pm} = \pm \sqrt{2} \mp i\sqrt{2} \) are not inside the curve \( C_r \) for any \( r > 0 \), but for \( r \) large enough, \( z_{\pm} = \pm \sqrt{2} + i\sqrt{2} \) are inside \( C_r \). Thus we have

\[ \lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = 2\pi i \left[ \text{Res}(\sqrt{2} + i\sqrt{2}) + \text{Res}(-\sqrt{2} + i\sqrt{2}) \right]. \]

That is,

\[ \int_{0}^{\infty} \frac{x^2}{x^4 + 16} \, dx = \pi i \left[ \text{Res}(\sqrt{2} + i\sqrt{2}) + \text{Res}(-\sqrt{2} + i\sqrt{2}) \right]. \]
Now we calculate the residue at \( z_+ = \sqrt{2} + i\sqrt{2} \):

\[
R(\sqrt{2} + i\sqrt{2}) = \lim_{z \to \sqrt{2} + i\sqrt{2}} \left[ (z - \sqrt{2} - i\sqrt{2}) \frac{z^2}{z^4 + 16} \right]
\]

\[
= \lim_{z \to \sqrt{2} + i\sqrt{2}} \left[ \frac{z^2}{(z - \sqrt{2} + i\sqrt{2})(z + \sqrt{2} - i\sqrt{2})(z + \sqrt{2} + i\sqrt{2})} \right]
\]

\[
= \frac{(\sqrt{2} + i\sqrt{2})^2}{(2i\sqrt{2})(2\sqrt{2} + i\sqrt{2})(\sqrt{2} + i\sqrt{2})}
\]

\[
= \frac{\sqrt{2} + i\sqrt{2}}{16i}.
\]

Next, the residue at \( z_- = -\sqrt{2} + i\sqrt{2} \) is

\[
R(-\sqrt{2} + i\sqrt{2}) = \lim_{z \to -\sqrt{2} + i\sqrt{2}} \left[ (z + \sqrt{2} - i\sqrt{2}) \frac{z^2}{z^4 + 16} \right]
\]

\[
= \lim_{z \to -\sqrt{2} + i\sqrt{2}} \left[ \frac{z^2}{(z - \sqrt{2} + i\sqrt{2})(z + \sqrt{2} - i\sqrt{2})(z + \sqrt{2} + i\sqrt{2})} \right]
\]

\[
= \frac{(-\sqrt{2} + i\sqrt{2})^2}{2(-\sqrt{2} + i\sqrt{2})(-2\sqrt{2})(2i\sqrt{2})}
\]

\[
= \frac{\sqrt{2} - i\sqrt{2}}{16i}.
\]

Thus,

\[
\int_{0}^{\infty} \frac{x^2}{x^4 + 16} \, dx = \pi i \left[ R(\sqrt{2} + i\sqrt{2}) + R(-\sqrt{2} + i\sqrt{2}) \right]
\]

\[
= 2\pi i \left( \frac{\sqrt{2} + i\sqrt{2}}{16i} + \frac{\sqrt{2} - i\sqrt{2}}{16i} \right)
\]

\[
= 2\pi i \left( \frac{2\sqrt{2}}{16i} \right)
\]

\[
= \frac{\pi \sqrt{2}}{8}.
\]

So,

\[
\int_{0}^{\infty} \frac{x^2}{x^4 + 16} \, dx = \frac{\pi \sqrt{2}}{8}.
\]

Wolfram Alpha answers \( \frac{\pi}{4\sqrt{2}} \), which is the same thing.
14.7.16. Evaluate

\[
\int_0^\infty \frac{x \sin x}{9x^2 + 4} \, dx
\]

For \( r > 0 \), let

\[ S_r = \{ re^{i\theta} \in \mathbb{C} \mid \theta \in [0, \pi] \} \]

be the semicircle of radius \( r \) in the first and second quadrants of the plane and

\[ I_r = \{ z \in \mathbb{C} \mid \text{Im}(z) = 0, \, |z| \leq r \} \]

be the real interval \([-r, r]\) embedded in the complex plane. Then \( C_r = S_r \cup I_r \) is a simple loop in \( \mathbb{C} \). Let \( f : \mathbb{C} \to \mathbb{C} \) be the function

\[ f(z) = \frac{ze^{iz}}{9z^2 + 4}. \]

Then

\[
\oint_{C_r} f(z) \, dz = \int_{I_r} f(z) \, dz + \int_{S_r} f(z) \, dz
\]

\[
= \int_{-r}^{r} f(x) \, dx + \int_{0}^{\pi} f(re^{i\theta})re^{i\theta} \, d\theta
\]

\[
= \int_{-r}^{r} \frac{xe^{ix}}{9x^2 + 4} \, dx + \int_{0}^{\pi} \frac{ir^2e^{2i\theta}e^{-re^{i\theta}}}{9r^2e^{2i\theta} + 4} \, d\theta
\]

\[
= \int_{-r}^{r} \frac{xe^{ix}}{9x^2 + 4} \, dx + \int_{0}^{\pi} \frac{ir^2e^{-2re^{i\theta}}}{9r^2e^{2i\theta} + 4} \, d\theta.
\]

As \( r \to \infty \), \( e^{-2re^{i\theta}} \) decays to 0 faster than \( ir^2/(9r^2e^{2i\theta} + 4) \) converges. Thus, we have

\[
\lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ \int_{-r}^{r} \frac{xe^{ix}}{9x^2 + 4} \, dx \right] + \lim_{r \to \infty} \left[ \int_{0}^{\pi} \frac{ir^2e^{-2re^{i\theta}}}{9r^2e^{2i\theta} + 4} \, d\theta \right]
\]

\[
= \int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2 + 4} \, dx + \pi \lim_{r \to \infty} \left[ \int_{0}^{\pi} \frac{ir^2e^{-2re^{i\theta}}}{9r^2e^{2i\theta} + 4} \, d\theta \right]
\]

\[
= \int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2 + 4} \, dx + \pi \left( \lim_{r \to \infty} \int_{0}^{\pi} \frac{ir^2e^{-2re^{i\theta}}}{9r^2e^{2i\theta} + 4} \, d\theta \right).
\]

By the Residue Theorem, we also have

\[
\lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ 2\pi i \sum_{i} R(z_i) \right]
\]

where \( z_i \) is a singularity of \( f(z) \) inside \( C_r \). The singularities of \( f \) are the roots of the polynomial \( 9x^2 + 4 \), which are \( z_{\pm} = \pm \frac{2}{3}i \). The singularity \( z_- = -\frac{2}{3}i \) is not inside the curve \( C_r \), for any \( r > 0 \), but for \( r \) large enough, the singularity \( z_+ = \frac{2}{3}i \) is inside \( C_r \). Thus we have

\[
\lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = 2\pi i R\left( \frac{2}{3}i \right) \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2 + 4} \, dx = 2\pi i R\left( \frac{2}{3}i \right).
\]
Now we calculate the residue at the singularity \( z_+ = \frac{2}{3}i \):

\[
R\left(\frac{2}{3}i\right) = \lim_{z \to \frac{2}{3}i} \left[ \left( z - \frac{2}{3}i \right) \frac{z e^{iz}}{9z^2 + 4} \right]
\]

\[
= \lim_{z \to \frac{2}{3}i} \left[ \frac{z e^{iz}}{9z + 6i} \right]
\]

\[
= \frac{2}{3}i e^{-2/3} \cdot \frac{1}{12i} = \frac{\pi}{9 e^{2/3}}.
\]

Thus

\[
\int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2 + 4} \, dx = 2\pi i R\left(\frac{2}{3}i\right) = 2\pi i \cdot \frac{1}{18 e^{2/3}} = \frac{\pi}{9 e^{2/3}}.
\]

Since \( e^{ix} = \cos x + i \sin x \), by linearity of the integral we have

\[
\frac{\pi}{9 e^{2/3}} i = \int_{-\infty}^{\infty} \frac{x e^{ix}}{9x^2 + 4} \, dx
\]

\[
= \int_{-\infty}^{\infty} \frac{x [\cos x + i \sin x]}{9x^2 + 4} \, dx
\]

\[
= \int_{-\infty}^{\infty} \frac{x \cos x}{9x^2 + 4} \, dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{9x^2 + 4} \, dx.
\]

Since the real and imaginary parts of the above equations must be equal, we have

\[
\int_{-\infty}^{\infty} \frac{x \sin x}{9x^2 + 4} \, dx = \frac{\pi}{9 e^{2/3}}.
\]

Furthermore, \( x \sin (x)/(9x^2 + 4) \) is an even function, so that

\[
\int_{0}^{\infty} \frac{x \sin x}{9x^2 + 4} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{9x^2 + 4} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\pi}{9 e^{2/3}} = \frac{\pi}{18 e^{2/3}}.
\]

Therefore,

\[
\int_{0}^{\infty} \frac{x \sin x}{9x^2 + 4} \, dx = \frac{\pi}{18 e^{2/3}}.
\]

Wolfram Alpha confirms.
14.7.30.  a. By the method of Example 2 evaluate
\[ \int_{0}^{\infty} \frac{1}{1 + x^4} \, dx. \]

b. Evaluate the same integral using tables or computer to get the indefinite integral; unless you are very careful you may get zero. Explain why.

c. Make the change of variables \( u = x^4 \) in the integral and evaluate the \( u \) integral using (7.4).

a. Let \( f : \mathbb{R} \to \mathbb{R} \) be the function given by \( f(x) = 1/(1 + x^4) \). Then \( f \) is an even function, so that for \( a > 0 \) we have
\[ \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx. \]

Now consider the extension of \( f \) into the complex plane. For some \( r > 0 \), let
\[ S_r = \{ re^{i\theta} \in \mathbb{C} \mid \theta \in [0, \pi] \} \]
be the semicircle of radius \( r \) in the first and second quadrants of the plane and
\[ I_r = \{ z \in \mathbb{C} \mid \text{Im}(z) = 0, \ |z| \leq r \} \]
be the real interval \([ -r, r ]\) embedded in the complex plane. Then \( C_r = S_r \cup I_r \) is a simple loop in \( \mathbb{C} \). Then
\[
\oint_{C_r} f(z) \, dz = \int_{I_r} f(z) \, dz + \int_{S_r} f(z) \, dz = \int_{-r}^{r} f(x) \, dx + \int_{0}^{\pi} f(re^{i\theta})ire^{i\theta} \, d\theta
\]
\[ = 2 \int_{0}^{r} \frac{1}{1 + x^4} \, dx + \int_{0}^{\pi} \frac{iire^{i\theta}}{1 + r^4e^{4i\theta}} \, d\theta. \]

As \( r \to \infty \) we have
\[
\lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ 2 \int_{0}^{r} \frac{1}{1 + x^4} \, dx \right] + \lim_{r \to \infty} \left[ \int_{0}^{\pi} \frac{iire^{i\theta}}{1 + r^4e^{4i\theta}} \, d\theta \right]
\]
\[ = 2 \int_{0}^{\infty} \frac{1}{1 + x^4} \, dx + \int_{0}^{\pi} \lim_{r \to \infty} \left[ \frac{iire^{i\theta}}{1 + r^4e^{4i\theta}} \right] \, d\theta
\]
\[ = 2 \int_{0}^{\infty} \frac{1}{1 + x^4} \, dx + \int_{0}^{\pi} 0 \, d\theta = 2 \int_{0}^{\infty} \frac{1}{1 + x^4} \, dx. \]

By the Residue Theorem, we also have
\[
\lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = \lim_{r \to \infty} \left[ 2\pi i \sum_{i} R(z_i) \right]
\]
where \( z_i \) is a singularity of \( f(z) \) inside \( C_r \). The singularities of \( f \) are the roots of the polynomial \( 1 + x^4 \), which are \( z_{\pm} = \pm \frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2} \). The singularities \( z_{\pm} = \pm \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \) are not inside the curve \( C_r \) for any \( r > 0 \), but for \( r \) large enough, \( z_{\pm} = \pm \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \) are inside
Thus we have
\[
\lim_{r \to \infty} \left[ \oint_{C_r} f(z) \, dz \right] = 2\pi i \left[ R \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) + R \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \right].
\]
That is,
\[
\int_0^\infty \frac{1}{1 + x^4} \, dx = \pi i \left[ R \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) + R \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \right].
\]
Now we calculate the residue at \( z_{++} = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \):
\[
R \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = \lim_{z \to \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}} \left[ \frac{\left( z - \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)}{1 + z^4} \right] \frac{1}{1 + z^4}
\]
\[
= \lim_{z \to \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}} \left[ \frac{1}{(z - \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2})(z + \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2})(z + \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2})} \right]
\]
\[
= \left( \frac{2i \frac{\sqrt{2}}{2}}{2 \frac{\sqrt{2}}{2}} \right) \frac{1}{2 \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)} = -2\sqrt{2} + 2i\sqrt{2}.
\]
Next, the residue at \( z_{-} = -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \) is
\[
R \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = \lim_{z \to -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}} \left[ \frac{\left( z + \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)}{1 + z^4} \right] \frac{1}{1 + z^4}
\]
\[
= \lim_{z \to -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}} \left[ \frac{z^2}{(z - \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2})(z - \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2})(z + \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2})} \right]
\]
\[
= \frac{1}{2 \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \left( -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)} = \frac{1}{2\sqrt{2} + 2i\sqrt{2}}.
\]
Thus,
\[
\int_0^\infty \frac{1}{1 + x^4} \, dx = \pi i \left[ R \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) + R \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \right]
\]
\[
= \pi i \left( \frac{1}{-2\sqrt{2} + 2i\sqrt{2}} + \frac{1}{2\sqrt{2} + 2i\sqrt{2}} \right)
\]
\[
= \frac{\pi}{2\sqrt{2}} \left( \frac{1}{1 + i} + \frac{1}{1 - i} \right)
\]
\[
= \frac{\pi}{2\sqrt{2}} \cdot \frac{2}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.
\]
So,
\[
\int_0^\infty \frac{1}{1 + x^4} \, dx = \frac{\pi}{2\sqrt{2}}.
\]
b. According to Wolfram Alpha, the integral in part a. is equal to \( \pi/(2\sqrt{2}) \), which is what we found. Also according to Wolfram Alpha, the indefinite integral

\[
\int \frac{1}{1 + x^4} \, dx
\]

is equal to

\[
\frac{1}{4\sqrt{2}} \left[ -\ln \left( x^2 - \sqrt{2}x + 1 \right) + \ln \left( x^2 + \sqrt{2}x + 1 \right) \\
-2\tan^{-1} \left( 1 - \sqrt{2}x \right) + 2\tan^{-1} \left( \sqrt{2}x + 1 \right) \right] + K.
\]

c. Equation (7.5) says

\[
\int_0^\infty \frac{r^{p-1}}{1+r} \, dr = \frac{\pi}{\sin \left( \pi p \right)}
\]

Let \( u = x^4 \). Then \( du = 4x^3 \, dx \), so that

\[
dx = \frac{1}{4} x^{-3} \, du = \frac{1}{4} u^{-3/4} \, du = \frac{1}{4} u^{1/4-1} \, du.
\]

Then by (7.5) we have

\[
\int_0^\infty \frac{1}{1 + x^4} = \frac{1}{4} \int_0^\infty \frac{u^{1/4-1}}{1+u} \, du = \frac{1}{4} \cdot \frac{\pi}{\sin \left( \pi/4 \right)} = \frac{1}{4} \cdot \frac{\pi}{\sqrt{2}/2} = \frac{\pi}{2\sqrt{2}}
\]
as expected.
Along $\overrightarrow{OP}$, $f(z)$ is real, so $\Delta \text{arg } f(z) = 0$.

For $P$ sufficiently large, $z^2$ dominates expression, $f(z) \sim z^2 = R e^{i \theta}$

along $\overrightarrow{QP}$. Along $\overrightarrow{QO} z = iy$. $f(iy) = -i(y^3 - y) + y^2$

$\arg f(iy) = \arctan \left( \frac{y^2}{y^3} \right)$. The image of $\overrightarrow{QO}$ starts in the 3rd quadrant with argument near $\frac{3\pi}{2}$. As $y$ decreases, the image stays in the 3rd quadrant until $y = 2$ when the real part becomes positive, crossing into the 4th quadrant. The imaginary part becomes positive at $y = 1$, then returns to the positive $x$-axis as $y \to 0$. Thus $\Delta \text{arg } f(z) = \frac{\pi}{2}$ along $\overrightarrow{QO}$.

$N = \frac{1}{2\pi} \Delta_c \text{arg } f(z) = \frac{1}{2\pi} \left( \frac{\pi}{2} + 3\frac{\pi}{2} + \frac{\pi}{2} \right) = 1$.

The root in the 1st quadrant, the conjugate root in the 4th quadrant, and the real root on the negative axis.

Along $\overrightarrow{OQ}$, $f(z)$ is real, so $\Delta \text{arg } f(z) = 0$.

Along $\overrightarrow{OP}$, $z = iy$, so $\arg f(z) = \arctan \left( \frac{-y^3 + 2y}{y^4 - 4y^2 + 3} \right)$. At 0, the image begins at 3. As $y$ increases, the image curve traces a path through the 1st quadrant until $y = 1$, then the image crosses to the 2nd quadrant. At $y = \sqrt{2}$, the image crosses from the 2nd quadrant into the 3rd quadrant. At $y = \sqrt{3}$, the image moves from the 3rd quadrant to the 4th quadrant. As $y$ increases, it stays in the 4th quadrant and

$\lim_{y \to 0} \arg f(z) \to 0$, completing one encirclement of the origin.

Along $\overrightarrow{PQ}$, $f(z) \sim z^4 = R^4 e^{i4\theta}$. As $\theta$ traverses $\frac{\pi}{2}$ radians ($\frac{\pi}{2} \leq \theta \leq \pi$), the image of $f(z)$ has $\Delta \text{arg } f(z) \approx 2\pi$. Thus, the total change in argument gives $N = \frac{1}{2\pi} \Delta_c \text{arg } f(z) = \frac{1}{2\pi} \left( \frac{\pi}{2} + 2\pi + 2\pi \right) = 2$.

It follows there are two roots in the 2nd quadrant, so by the form of $f(z)$ there must also be two roots in the 3rd quadrant (complex conjugates). No roots appear in the 1st or 4th quadrants.