I, _____(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

1. Consider the heat equation in an insulated one-dimensional rod given by:

$$\frac{\partial u}{\partial t} = 0.2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 4, \quad t > 0,$$

with the boundary conditions and initial condition:

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(4,t) = 0, \quad u(x,0) = 5 - 3\cos(\pi x).$$

Solve this initial-boundary value problem. Find the eigenvalues and eigenfunctions for the associated Sturm-Liouville problem. What is the temperature distribution in the rod as $t \to \infty$?

2. If convection is taken into account, the equation for heat conduction and convection in a one-dimensional rod is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}, \quad 0 < x < L, \quad t > 0.$$

Let $k = 1$, $v_0 = 0.5$, and $L = 10$. Assume the following boundary conditions and initial conditions:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

a. Use separation of variables to create two ordinary differential equations.

b. From the spatial ordinary differential equation, create a Sturm-Liouville eigenvalue problem of the form:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0.$$

Identify explicitly the functions $p(x)$, $q(x)$, and $\sigma(x)$. Find the eigenvalues and eigenfunctions for this problem.

c. Solve the original partial differential equation with its boundary and initial conditions. Write clearly your integral for finding the Fourier coefficients.

3. a. The displacement of a uniform thin beam in a medium that resists motion satisfies the beam equation:

$$\frac{\partial^4 u}{\partial x^4} = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - k\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0,$$

where $k < \frac{2\pi^2}{a^2 c}$. If the beam is simply supported at the ends, then the boundary conditions are:

$$u(0,t) = 0, \quad u_{xx}(0,t) = 0, \quad u(a,t) = 0, \quad u_{xx}(a,t) = 0.$$

Assume that there is initially no displacement and that an initial velocity, $u_t(x,0) = 1$ is given to the beam. Solve this initial-boundary value problem. You can assume that the eigenvalues are real, but show clearly how you obtain all eigenvalues and eigenfunctions.

b. Let $a = 2$, $c = 1$, and $k = 0.1$. Use 20 terms in the series solution of $u(x,t)$ and have the computer graph the displacement of the beam at times $t = 0$, 1, 2, 5, 10, and 20.

4. Consider heat conduction in a sphere given by:

$$\frac{\partial u}{\partial t} = \frac{k}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right), \qquad 0 < \rho < a, \quad t > 0,$$

with the boundary and initial conditions:

$$u(a,t) = 0, \quad u(\rho,0) = T_0.$$

Solve this equation noting any other boundary conditions you might need to apply. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. (Hint: You might want to try the change of variables given by $u(\rho,t) = v(\rho,t)/\rho$.)

5. Find the steady-state temperature in a cube, which satisfies:

$$\nabla^2 u(x,y,z) = 0, \qquad 0 < x < 2, \quad 0 < y < 2, \quad 0 < z < 2.$$

The cube is insulated on the faces with $x = 0$ and $y = 2$. The cube is kept at 0°C on the faces with $x = 2$ and $z = 0$ and kept at T_0 when $y = 0$. Finally, it satisfies Newton's law of cooling on the other face ($z = 2$) with

$$-k\frac{\partial u(x,y,2)}{\partial z} = hu(x,y,2).$$

6. A can of beer at room temperature (20°C) is almost submersed in ice water (0°C).

a. Find the steady-state temperature of the beer assuming it satisfies Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \qquad 0 < r < 1, \quad 0 < z < 4,$$

with the boundary conditions:

$$u(1,z) = 0, \quad u(r,0) = 0, \quad u(r,4) = 20.$$

You can assume there is infinite ice. For **extra-credit**, assume that you pour this beer into a glass (beer becomes well-mixed, so takes the average steady-state temperature), don't assume any heat transfer from the glass, then use 50 terms in your solution to determine what is the average temperature of the beer.

b. In this part of the problem, we want to know the time evolution of the cooling of the beer. The can of beer satisfies the heat equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \qquad 0 < r < 1, \quad 0 < z < 4, \quad t > 0,$$

with the boundary conditions:

$$u(1, z, t) = 0, \quad u(r, 0, t) = 0, \quad u(r, 4, t) = 20,$$

and initial condition:

$$u(r, z, 0) = 20.$$

Find the temperature of the beer, $u(r, z, t)$ for all $t > 0$. (Hint: You may want to take advantage of the steady-state solution in Part a to create homogeneous boundary conditions for this part of the problem.)