Calculus for the Life Sciences I
Lecture Notes – Discrete Malthusian Growth

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Outline

1. Population of the United States
   - Census Data
   - Growth Rate
   - Malthusian Growth Model

2. Discrete Malthusian Growth
   - Solution of Malthusian Growth Model

3. Compound Interest

4. Discrete Population Models
   - Autonomous
   - Nonautonomous

5. U. S. Population Modeling
   - Discrete Malthusian Growth
   - Varying Growth Rate
United States Census

- Constitution requires census every 10 years
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- Census used for budgeting federal payments and representation in Congress
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- Constitution requires census every 10 years
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- Process can be politically charged
- Accurately predicting demographic data are important for planning communities in the future
- Calculations for the future populations uses sophisticated mathematical models
- Models are constantly improved and revised
### Census Data

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>3,929,214</td>
</tr>
<tr>
<td>1800</td>
<td>5,308,483</td>
</tr>
<tr>
<td>1810</td>
<td>7,239,881</td>
</tr>
<tr>
<td>1820</td>
<td>9,638,453</td>
</tr>
<tr>
<td>1830</td>
<td>12,866,020</td>
</tr>
<tr>
<td>1840</td>
<td>17,069,453</td>
</tr>
<tr>
<td>1850</td>
<td>23,191,876</td>
</tr>
<tr>
<td>1860</td>
<td>31,443,321</td>
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<tr>
<td>1870</td>
<td>39,818,449</td>
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<td>1880</td>
<td>50,189,209</td>
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<tr>
<td>1890</td>
<td>62,947,714</td>
</tr>
<tr>
<td>1900</td>
<td>76,212,168</td>
</tr>
<tr>
<td>1910</td>
<td>92,228,496</td>
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<tr>
<td>1920</td>
<td>106,021,537</td>
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<tr>
<td>1930</td>
<td>122,775,046</td>
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<tr>
<td>1940</td>
<td>132,164,569</td>
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<tr>
<td>1950</td>
<td>150,697,361</td>
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<tr>
<td>1960</td>
<td>179,323,175</td>
</tr>
<tr>
<td>1970</td>
<td>203,302,031</td>
</tr>
<tr>
<td>1980</td>
<td>226,545,805</td>
</tr>
<tr>
<td>1990</td>
<td>248,709,873</td>
</tr>
<tr>
<td>2000</td>
<td>281,421,906</td>
</tr>
<tr>
<td>2010</td>
<td>308,745,538</td>
</tr>
</tbody>
</table>
Growth Rate in Early U. S.
The growth rate for the decade of 1790-1800

\[
\frac{\text{Population in 1800}}{\text{Population in 1790}} = \frac{5,308,483}{3,292,214} = 1.351
\]
Growth Rate of U. S.

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The growth rate for the decade of 1800-1810

$$\frac{\text{Population in 1810}}{\text{Population in 1800}} = \frac{7,239,881}{5,308,483} = 1.364$$
Growth Rate of U. S.

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The growth rate for the decade of 1790-1800

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\]

The growth rate for the decade of 1810-1820

\[
\frac{\text{Population in 1820}}{\text{Population in 1810}} = \frac{9,638,453}{7,239,881} = 1.331
\]
The growth rates for the decades following 1790, 1800, and 1810 are 35.1%, 36.4%, and 33.1%.

The average is 34.9% per decade.
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This growth rate remains almost constant until 1860
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The average is 34.9% per decade
This growth rate remains almost constant until 1860
Suggests a constant growth rate model
Malthusian Growth Model
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Simplest growth model uses a constant rate of growth, $r$. 
Malthusian Growth Model

- Simplest growth model uses a constant rate of growth, $r$
- Start with the population in 1790, $P_0$
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- Start with the population in 1790, \( P_0 \)
- Population in the next decade is current population plus the population times the average growth rate

\[
P_{n+1} = P_n + rP_n = (1 + r)P_n
\]
Malthusian Growth Model

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- Start with the population in 1790, $P_0$
- Population in the next decade is current population plus the population times the average growth rate

$$P_{n+1} = P_n + rP_n = (1 + r)P_n$$

- Sequence of predicted populations based solely on population from preceding population
Malthusian Growth Model for U. S. Population (early years)
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Let $P_0 = 3,929,214$ (population 1790) and take $r = 0.349$
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Malthusian Growth Model for U. S. Population (early years)
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For 1800, model gives

$$P_1 = 1.349P_0 = 5,300,510$$
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For 1810, model gives

$$P_2 = 1.349P_1 = 7,150,388$$
Malthusian Growth Model for U. S. Population (early years)
Let \( P_0 = 3,929,214 \) (population 1790) and take \( r = 0.349 \)
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P_2 = 1.349P_1 = 7,150,388
\]
For 1820, model gives
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P_3 = 1.349P_2 = 9,645,873
\]
Table for U. S. Population (early years)

<table>
<thead>
<tr>
<th>Year</th>
<th>Census</th>
<th>Model $P_{n+1} = 1.349P_n$</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>3,929,214</td>
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<td></td>
</tr>
<tr>
<td>1800</td>
<td>5,308,483</td>
<td>5,300,510</td>
<td>−0.15</td>
</tr>
<tr>
<td>1810</td>
<td>7,239,881</td>
<td>7,150,388</td>
<td>−1.24</td>
</tr>
<tr>
<td>1820</td>
<td>9,638,453</td>
<td>9,645,873</td>
<td>0.08</td>
</tr>
<tr>
<td>1830</td>
<td>12,866,020</td>
<td>13,012,282</td>
<td>1.14</td>
</tr>
<tr>
<td>1840</td>
<td>17,069,453</td>
<td>17,553,569</td>
<td>2.84</td>
</tr>
<tr>
<td>1850</td>
<td>23,191,876</td>
<td>23,679,765</td>
<td>2.10</td>
</tr>
<tr>
<td>1860</td>
<td>31,433,321</td>
<td>31,944,002</td>
<td>1.59</td>
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<tr>
<td>1870</td>
<td>39,818,449</td>
<td>43,092,459</td>
<td>8.22</td>
</tr>
</tbody>
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Malthusian Growth Model

Graph of the **Malthusian Growth Model** and **Census Data** for the U. S.

![Graph showing U.S. Population over time.](image)
Early constant growth rate of about 35% 

Error remains small until 1870 because of the fairly constant rate of growth (Agrarian society)
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- Error remains small until 1870 because of the fairly constant rate of growth (Agrarian society)
- Most predicted populations are a little high, suggesting the 19th century growth rate declined
- Civil War created dramatic decline in the growth rate
- More significantly, population demographics changed as the U. S. moved into the industrial revolution away from agriculture
Changing Growth Rate

Variation in Growth Rate

- Assume this Malthusian growth model were extended
Changing Growth Rate

**Variation in Growth Rate**

- Assume this Malthusian growth model were extended
  - In 1920, model predicts 192,365,343 (population in 1960s), which is 82% too high
  - In 1970, model predicts 859,382,645, which is 323% too high
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- Average growth rate over census history is 22.3%
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- Average growth rate over census history is 22.3%
- Growth rate in 1920 is 15%, dropping to 13% in 1970
- Lowest growth rate during the Great Depression of 7.2%
- Latest growth rate for U. S. is 9.7%
Discrete Malthusian Growth Model

\[ P_{n+1} = P_n + rP_n = (1 + r)P_n, \]

where \( r \) is the average growth rate.
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Next generation is proportional to the population of the current generation
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Next generation is proportional to the population of the current generation

- Named for Thomas Malthus (1766-1834)
- Example of Discrete Dynamical system or Difference Equations
- Population models using difference equations are common in ecological models
The **Malthusian growth model** is one of few easily solved discrete models
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\[ P_1 = (1 + r)P_0 \]
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P_n = (1 + r)P_{n-1} = \ldots = (1 + r)^nP_0
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\[ P_2 = (1 + r)P_1 = (1 + r)^2P_0 \]

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General solution is given by

\[ P_n = (1 + r)^nP_0 \]
General solution of Malthusian growth model

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This solution shows why Malthusian growth is also known as exponential growth.

The solution to the model is an exponential function with a base of \(1 + r\) and power \(n\) representing the number of iterations after the initial population.
Example – Malthusian Growth

Suppose that a population of yeast, satisfying Malthusian growth, grows 10% in an hour. If the population begins with 100,000 yeast, then find the population at the end of 4 hours.

How long does it take for this population to double?
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The general solution is

\[ P_n = (1.1)^n P_0 = 100,000 (1.1)^n \]
Solution (cont): The population after 4 hours

\[ P_4 = 100,000(1.1)^4 = 146,410 \]
Example – Malthusian Growth

Solution (cont): The population after 4 hours

\[ P_4 = 100,000(1.1)^4 = 146,410 \]

For the solution to double

\[ 200,000 = 100,000(1.1)^n \quad \text{or} \quad (1.1)^n = 2 \]
Solution (cont): The population after 4 hours

\[ P_4 = 100,000(1.1)^4 = 146,410 \]

For the solution to double

\[ 200,000 = 100,000(1.1)^n \quad \text{or} \quad (1.1)^n = 2 \]

Taking logarithms

\[ n \ln(1.1) = \ln(2) \quad \text{or} \quad n = \frac{\ln(2)}{\ln(1.1)} = 7.27 \text{ hr} \]
Population Studies - Discrete Malthusian Growth

a. One species of insect grows according to the discrete Malthusian growth model

\[ H_{n+1} = 1.06H_n, \quad H_0 = 50,000 \]

where \( n \) is in weeks

Find the population at the end of the first three weeks, \( H_1, H_2, \) and \( H_3 \)

Determine how long it takes for this population to double
Solution a: The Malthusian growth model satisfies

\[ H_n = (1.06)^n H_0 = 50,000(1.06)^n \]
Example – Two Populations

**Solution a:** The Malthusian growth model satisfies

\[ H_n = (1.06)^n H_0 = 50,000(1.06)^n \]

It follows that

\[ H_1 = 50,000(1.06) = 53,000 \quad H_2 = 56,180 \quad H_3 = 59,551 \]
Example – Two Populations

Solution a: The Malthusian growth model satisfies

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It follows that

\[ H_1 = 50,000 (1.06) = 53,000 \quad H_2 = 56,180 \quad H_3 = 59,551 \]

The doubling time

\[ 2H_0 = (1.06)^n H_0 \]
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\[ H_1 = 50,000 (1.06) = 53,000 \quad H_2 = 56,180 \quad H_3 = 59,551 \]

The doubling time

\[ 2H_0 = (1.06)^n H_0 \]

With logarithms

\[ \ln(2) = n \ln(1.06) \quad \text{or} \quad n = \frac{\ln(2)}{\ln(1.06)} = 11.90 \text{ weeks} \]
b. Another insect species starts with a smaller population, but grows more quickly

\[ G_{n+1} = 1.08G_n, \quad G_0 = 10,000 \]

Find the doubling time of this population of insects

Determine how long until the populations of the two species are equal
Example – Two Populations

b. Another insect species starts with a smaller population, but grows more quickly

\[ G_{n+1} = 1.08G_n, \quad G_0 = 10,000 \]

Find the doubling time of this population of insects

Determine how long until the populations of the two species are equal

**Solution b:** This population satisfies

\[ G_n = (1.08)^n G_0 = 10,000(1.08)^n \]
Example – Two Populations

b. Another insect species starts with a smaller population, but grows more quickly

\[ G_{n+1} = 1.08G_n, \quad G_0 = 10,000 \]

Find the doubling time of this population of insects

Determine how long until the populations of the two species are equal

**Solution b:** This population satisfies

\[ G_n = (1.08)^n G_0 = 10,000(1.08)^n \]

The doubling time satisfies

\[ \ln(2) = n \ln(1.08) \quad \text{or} \quad n = \frac{\ln(2)}{\ln(1.08)} = 9.0 \text{ weeks} \]
Solution b (cont): The two populations are equal when

\[
(1.08)^n G_0 = (1.06)^n H_0 \\
10,000(1.08)^n = 50,000(1.06)^n \\
\left(\frac{1.08}{1.06}\right)^n = 5 \\
n \ln \left(\frac{1.08}{1.06}\right) = \ln(5) \\
n = 86.1 \text{ weeks}
\]
Solution b (cont): The two populations are equal when

\[
\begin{align*}
(1.08)^n G_0 &= (1.06)^n H_0 \\
10,000(1.08)^n &= 50,000(1.06)^n \\
\left(\frac{1.08}{1.06}\right)^n &= 5 \\
n \ln \left(\frac{1.08}{1.06}\right) &= \ln(5) \\
n &= 86.1 \text{ weeks}
\end{align*}
\]

The two populations are approximately equal after 86 weeks.
Compound Interest

Compound interest problems are closely related to Malthusian growth models.
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Start with an initial principal $P_0$ and an annual interest rate of $r$
Compound interest problems are closely related to Malthusian growth models.

Start with an initial principal $P_0$ and an annual interest rate of $r$.

The principal $n$ years later, $P_n$ satisfies

$$P_{n+1} = (1 + r)P_n \quad \text{given } P_0$$

or

$$P_n = (1 + r)^n P_0$$
When interest is compounded \( k \) times a year, the formula for the amount of principal, \( P_n \), given an initial principal \( P_0 \) and an annual interest rate of \( r \) satisfies

\[
P_n = \left( 1 + \frac{r}{k} \right)^{kn} P_0
\]

where \( n \) is in years.
Example: Suppose you begin with $2,000 to invest. **Bank A** offers 6.25% interest compounded annually, while **Bank B** offers 6% interest compounded monthly. Which of these investments gives the better return?
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Solution: Using the model above for Bank A, we have $k = 1$, $r = 0.0625$, and $P_0 = $2,000, so after a year

$$P_{1A} = (1 + 0.0625)^1($2,000) = $2,125$$
Solution (cont): For Bank B, $k = 12$, $r = 0.06$, and $P_0$ is also $2,000$, so after one year

$$P_{1B} = \left(1 + \frac{0.06}{12}\right)^{12} \times 2,000 = 2,123.36$$
Solution (cont): For Bank B, \( k = 12 \), \( r = 0.06 \), and \( P_0 \) is also $2,000, so after one year

\[
P_{1B} = \left(1 + \frac{0.06}{12}\right)^{12} (\$2,000) = \$2,123.36
\]

So Bank A has a slightly better return by $1.64
The population data for 1790 is 3,929,214, while the population data for 1800 is 5,308,483
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This gives a decade growth rate of 35.1%
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What is the annual growth rate?
The population data for 1790 is 3,929,214, while the population data for 1800 is 5,308,483.

This gives a decade growth rate of 35.1%.

What is the annual growth rate?

**Solution:** If we let $n$ be in years, then to find the annual growth rate, we solve

$$P_{10} = (1 + r)^{10}P_0$$
Solution (cont): Solve

\[
5,308,483 = (1+r)^{10}3,929,214
\]
\[
(1+r)^{10} = 1.351
\]
\[
1+r = 1.351^{1/10} = 1.03054
\]
\[
r = 0.03054
\]
Solution (cont): Solve

\[
\begin{align*}
5,308,483 &= (1 + r)^{10} 
\times 3,929,214 \\
(1 + r)^{10} &= 1.351 \\
1 + r &= 1.351^{1/10} = 1.03054 \\
r &= 0.03054
\end{align*}
\]

It follows that annual growth rate is \( r = 0.03054 \) or 3.054%
Solution (cont): Solve

\[ 5,308,483 = (1 + r)^{10} \times 3,929,214 \]
\[ (1 + r)^{10} = 1.351 \]
\[ 1 + r = 1.351^{1/10} = 1.03054 \]
\[ r = 0.03054 \]

It follows that annual growth rate is \( r = 0.03054 \) or 3.054%.

Note that decade growth was 35.1%, which is more than 10 times the annual growth rate.

This shows the effects of compounding interest.
Example The population in the U. S. was 203.3 million in 1970 and 226.5 million in 1980.

Assume that the population is growing according to the discrete Malthusian growth model and find the annual growth rate of the population during this period of time.
Example The population in the U. S. was 203.3 million in 1970 and 226.5 million in 1980.

Assume that the population is growing according to the discrete Malthusian growth model and find the annual growth rate of the population during this period of time.

Use this information to project the population in 1990.

The actual census gives the population in 1990 to be 248.7 million, so what is the percent error between the actual population and the modeling prediction?
Example for Population Growth

**Solution:** Let \( P_0 = 203.3 \) and \( P_{10} = 226.5 \)
Example for Population Growth

**Solution:** Let $P_0 = 203.3$ and $P_{10} = 226.5$

The decade growth rate satisfies:

$$\frac{P_{10}}{P_0} = \frac{226.5}{203.3} = 1.1141 = 1 + r_d$$

Thus, the growth rate per decade in 1970 was 11.41%
Example for Population Growth

**Solution:** Let \( P_0 = 203.3 \) and \( P_{10} = 226.5 \)

The decade growth rate satisfies:

\[
\frac{P_{10}}{P_0} = \frac{226.5}{203.3} = 1.1141 = 1 + r_d
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Thus, the growth rate per decade in 1970 was 11.41%

The annual growth rate satisfies:

\[
203.3(1 + r_a)^{10} = 226.5 \\
(1 + r_a)^{10} = 1.1141 \\
1 + r_a = 1.1141^{1/10} = 1.01086
\]
Example for Population Growth

Solution: Let $P_0 = 203.3$ and $P_{10} = 226.5$

The decade growth rate satisfies:

$$\frac{P_{10}}{P_0} = \frac{226.5}{203.3} = 1.1141 = 1 + r_d$$

Thus, the growth rate per decade in 1970 was 11.41%

The annual growth rate satisfies:

$$203.3(1 + r_a)^{10} = 226.5$$

$$(1 + r_a)^{10} = 1.1141$$

$$1 + r_a = 1.1141^{1/10} = 1.01086$$

The annual growth rate is $r_a = 0.01086$ or 1.086%
Example for Population Growth

Solution (cont): The discrete Malthusian growth model is

\[ P_n = (1.01086)^n P_0 = 203.3(1.01086)^n \]

where \( n \) is in years.
Solution (cont): The discrete Malthusian growth model is

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For \( n = 20 \) years in 1990, we obtain a population of

\[ P_{20} = 203.3(1.01086)^{20} = 252.3 \text{ million} \]
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For \( n = 20 \) years in 1990, we obtain a population of

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The actual census is 248.7 million, so the percent error of this model is

\[
100 \left( \frac{252.3 - 248.7}{248.7} \right) = 1.45\%
\]
The general **Discrete Dynamical Population Model** (time-independent)

\[ P_{n+1} = f(P_n) \]
Discrete Population Models

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This difference equation is **Autonomous**, since the function \( f \) depends only on the population.
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A more general **Discrete Dynamical Population Model** with **temporal** or **time dependence**

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The general **Discrete Dynamical Population Model** 
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This difference equation is **Nonautonomous**
The average growth rate for U. S. over its history

\[ r = 0.2233 \]
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The best discrete Malthusian growth model is

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The average growth rate for U. S. over its history

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The best discrete Malthusian growth model is

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This growth rate is too low for the early years, and too high for later years.
A **modified time dependent growth rate** is found by fitting a line through the data from 1790 to 1990.

\[ k(t) = 0.3744 - 0.001439t \]

**Growth Rate/Decade for U. S.**

- **Average Growth Rate**
- **Years After 1790**
- **Growth Rate**
The best fit to the growth data from 1790 to 1990 satisfies

\[ k(t) = 0.3744 - 0.001439 t \]

where \( t \) is the number of years after 1790.
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The Nonautonomous Malthusian Growth Model satisfies

\[ P_{n+1} = (1 + k(t_n))P_n \]

where \( t_n = 10n \) and \( n \) is the number of decades after 1790
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The Nonautonomous Malthusian Growth Model satisfies

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where \( t_n = 10n \) and \( n \) is the number of decades after 1790.

The model can be written

$$P_{n+1} = (1.3744 - 0.01439 n)P_n$$
Graph of the **Discrete Malthusian Growth Model** and **Nonautonomous Discrete Malthusian Growth Model** for the U. S. Population (with both models starting $P_0 = 3,929,214$)

![Graph of population growth models](image)
The constant growth rate discrete Malthusian growth model does poorly over this long period of time.
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- The maximum error occurs in 1950 with 11.7% error.
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Both models over predict the 2010 census.
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- The average absolute percent error is only 5.1%.
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Both models over predict the 2010 census.

- The discrete Malthusian growth model predicts a population of 331,214,433.
- The nonautonomous discrete Malthusian growth model predicts a population of 311,407,591.

These produce 7.3% and 0.9% errors, respectively.
Example A population of arthropods is growing in a lake that begins to receive pesticide runoff from neighboring farm fields. The resulting pollution adversely affects the rate of growth of their population.
Example A population of arthropods is growing in a lake that begins to receive pesticide runoff from neighboring farm fields. The resulting pollution adversely affects the rate of growth of their population.

Suppose the nonautonomous Malthusian growth model for the arthropods is

\[ A_{n+1} = (1 + k(t_n))A_n \quad A_0 = 200(\text{per l}^3) \]

where \( n \) is weeks, \( k(t_n) = 0.1 - 0.02n \), and \( A_n \) is the population density after \( n \) weeks.
Example of Nonautonomous Growth

For the nonautonomous Malthusian growth model

\[ A_{n+1} = (1.1 - 0.02n)A_n \quad A_0 = 200 \]

- Find the population at the end of the first three weeks, \( A_1 \), \( A_2 \), and \( A_3 \)
Example of Nonautonomous Growth

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- Find the maximum population density of these arthropods and when this occurs
Example of Nonautonomous Growth

For the nonautonomous Malthusian growth model

$$A_{n+1} = (1.1 - 0.02n)A_n \quad A_0 = 200$$

- Find the population at the end of the first three weeks, $A_1$, $A_2$, and $A_3$
- Find the maximum population density of these arthropods and when this occurs
- Determine when the lake becomes so polluted that the arthropod population dies out
Example of Nonautonomous Growth

**Solution:** Unfortunately, this nonautonomous growth model does NOT have a general solution, like the Malthusian growth model above.
Example of Nonautonomous Growth

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The first three weeks, $A_1$, $A_2$, and $A_3$, are found by simulation:

\[
A_1 = (1 + (0.1 - 0.02(0)))^{200} = (1.1)^{200} = 220,
\]
\[
A_2 = (1 + (0.1 - 0.02(1)))^{220} = (1.08)^{220} = 237.6,
\]
\[
A_3 = (1 + (0.1 - 0.02(2)))^{237.6} = (1.06)^{237.6} = 252.86.
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Finding when the maximum density occurs is easy, as it will occur when the growth rate falls to zero.
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\end{align*}
\]

Finding when the maximum density occurs is easy, as it will occur when the growth rate falls to zero

\[
k(t_n) = 0.1 - 0.02n = 0
\]

which happens at $n_{max} = 5$.
**Solution (cont):** Since there is no general solution, knowing when the maximum occurs only tells how far we need to simulate.
Example of Nonautonomous Growth

Solution (cont): Since there is no general solution, knowing when the maximum occurs only tells how far we need to simulate.

Below are simulations for 10 weeks (which is easily done in Excel)

<table>
<thead>
<tr>
<th>Week</th>
<th>Arthropods</th>
<th>Week</th>
<th>Arthropods</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200</td>
<td>6</td>
<td>267</td>
</tr>
<tr>
<td>1</td>
<td>220</td>
<td>7</td>
<td>262</td>
</tr>
<tr>
<td>2</td>
<td>238</td>
<td>8</td>
<td>251</td>
</tr>
<tr>
<td>3</td>
<td>252</td>
<td>9</td>
<td>236</td>
</tr>
<tr>
<td>4</td>
<td>262</td>
<td>10</td>
<td>217</td>
</tr>
<tr>
<td>5</td>
<td>267</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Solution (cont): Theoretically, the arthropod population dies out when \( 1 + k(t_n) = 0 \)

\[
1.1 - 0.02n = 0 \quad \text{or} \quad n = 55 \text{ weeks}
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Example of Nonautonomous Growth

Solution (cont): Theoretically, the arthropod population dies out when $1 + k(t_n) = 0$

$$1.1 - 0.02n = 0 \quad \text{or} \quad n = 55 \text{ weeks}$$

Numerical simulations show that this population drops below 1 arthropod/\text{l}^3 in only 28 weeks
**Example of Nonautonomous Growth**

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- From week 28 to 55, the population is very small
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- Numerical simulations show that this population drops below 1 arthropod/l^3 in only 28 weeks
- From week 28 to 55, the population is very small
- Practically speaking, this population is extinct after the 28\(^{th}\) week
- There is some discrepancy between theoretical and numerical extinction with this more complicated model