Spring 2007 Math 336 Name________________________________

I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

Give all answers to at least **4 significant figures**. Be sure to include appropriate **units** for each of your answers.

1. Michael Crichton in the *Andromeda Strain* (1969) states that "A single cell of the bacterium *E. coli* would, under ideal circumstances, divide every twenty minutes... [I]t can be shown that in a single day, one cell of *E. coli* could produce a super-colony equal in size and weight to the entire planet Earth." A single *E. coli* has a volume of about 1.7 μm^3. The diameter of the Earth is 12,756 km, so assuming it is a perfect sphere, determine how long it takes for an ideally growing colony (Malthusian growth) of *E. coli* (doubling every 20 min) to equal the volume of the Earth.

2. When a monoculture of an organism is grown in a limited (but renewed) medium, then the population of that organism often follows the logistic growth model. Below is a table for the growth of a fresh water organism.

Day	0	2	3	5	7	9	11	12	15
Population(/cc)	2	5	9	27	63	109	186	211	227

a. Consider the discrete logistic growth model

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right), \qquad P(0) = P_0.$$

Use the data above to find the parameters P_0, r, and M that best fit this model. Write the least sum of square errors. Briefly discuss the meaning of each of the parameters and what each term in the model represents with respect to population dynamics.

b. Find all equilibria for the logistic growth model with the parameters found above. Find the derivative of the updating function and evaluate it at each of the equilibria. Use this information to give the stability of all the equilibria. What is the carrying capacity for this model?

c. Another important model used in ecology is the Beverton-Holt model given by

$$P_{n+1} = \frac{\mu K P_n}{K + (\mu - 1)P_n}, \qquad P(0) = P_0.$$

Use the data above to find the parameters P_0, μ, and K that best fit this model. Write the least sum of square errors. Briefly discuss what each of these parameters represent and compare them to the parameters in the logistic growth model.

d. Find all equilibria for the Beverton-Holt growth model with the parameters found above. Find the derivative of the updating function and evaluate it at each of the equilibria. Use this information to give the stability of all the equilibria. What is the carrying capacity for this model?

e. Graph the time series for both models and the data. Be sure to have the models as lines in your graph and data as points (all clearly labeled).

f. Graph the updating functions for both of these models. (Graph $P_n \in [0, 300]$.) Include the identity map in your graph. Qualitatively, what are the main differences between each of these updating functions? (Hint: consider larger populations than shown on your graph.) Are there significant differences in the domain where the data are collected?

g. Which model fits the data best or are the models very similar? Give a reason why ecologists might prefer the Beverton-Holt model to the logistic model for population studies. List at least one strength of each model. List at least one weakness with each model.

3. a. Suppose that a population of fish, $F(t)$ (in thousands), satisfies the logistic growth model given by

$$\frac{dF}{dt} = 0.4\,F\left(1 - \frac{F}{200}\right),$$

where t is in years. Find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait on the F-axis. (Use open circles for unstable equilibria and closed circles for stable equilibria.) What is the stability of each of the equilibria? Determine the carrying capacity for this population of fish.

b. Assume that fishing is allowed and that 15,000 fish are harvested annually. The model becomes

$$\frac{dF}{dt} = 0.4\,F\left(1 - \frac{F}{200}\right) - 15.$$

Find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait on the F-axis. What is the stability of each of the equilibria? Determine the carrying capacity for this population of fish. What is the threshold number of fish needed to avoid extinction?

c. If the model for fishing with a harvesting level of h (in thousands) is given by

$$\frac{dF}{dt} = 0.4\,F\left(1 - \frac{F}{200}\right) - h,$$

then determine the maximum level of harvesting that allows the fish population to exist without going to extinction. Draw a bifurcation diagram for $h \geq 0$, showing the stable equilibria with solid lines and unstable equilibria with dashed (or dotted) lines. As h increases, what value of h causes a qualitative change in the model and what type of bifurcation is this?

4. The text used a nonstandard dart board for its homework problem of darts. Below is a picture of the standard dart board. Assume that the dartboard has a radius of 12 inches. Assume that starting from the center, the bull's eye (red) has a radius of 0.5 inch and is worth 50 points and the next annular region (green) has a radius of 0.5 inch and is worth 25 points. The black and white sectors are worth the number of points labeled on the outside ring, ranging from 1 to 20 points. The inner of these black and white sectors is an annulus with a radius of 4 inches, and the outer black and white annulus has a radius of 3 inches. The ring between these two annuli (green and red) is a triple score region (three times the score listed in the outside region) and has a radius of 0.5 inches. The outer ring of green and red regions is a double score region with a radius of 0.5 inches. The outer most region has a radius of 3 inches and is worth no points. We assume that each of the 20 sectors has equal area. You might note that the arrangement

of the sectors has low value regions next to high value regions, adding significantly to the skill required for hitting the region desired. You may want to learn the various versions of the game.

a. A standard round of darts has the person throw three darts. Assume that all darts hit the dart board randomly. What is the highest possible score, and how is that achieved? Design a Monte Carlo simulation for scoring three darts with the assumption of hitting the board at random. List 5 sets of three darts thrown giving the score of each dart and the total for each of the 5 rounds.

b. Simulate your program 10,000 times to obtain the mean score of throwing three darts and the standard deviation of the score for your 10,000 rounds of three darts.

5. An age-structured population of birds was surveyed over 4 years. The researchers determined the number of birds in each age class for each of the 4 years and found out how many nestlings fledged from each of the different age classes each year. The researchers divided the population of birds into the birds 0-1 years old, 1-2 years old, and those that are older. This age-structured population forms a Leslie model of the following form:

$$\begin{pmatrix} P_1(n+1) \\ P_2(n+1) \\ P_3(n+1) \end{pmatrix} = \begin{pmatrix} 0 & b_2 & b_3 \\ s_{12} & 0 & 0 \\ 0 & s_{23} & s_{33} \end{pmatrix} \begin{pmatrix} P_1(n) \\ P_2(n) \\ P_3(n) \end{pmatrix}.$$

a. The table below shows how many birds in each age class survived to the next year (and gives the total number of birds that fledged). The researchers determined that the survival of the 1-2 year old birds is roughly equal to the survival of the older birds. Thus, we can assume that $s_{23} = s_{33}$. Use the data below to compute the average values for each of the survival parameters s_{12} and $s_{23} = s_{33}$.

Bird Age	Year 1	Year 2	Year 3	Year 4
0-1	175	237	258	311
1-2	42	59	89	92
older	97	104	128	145

They also collected data on the success rate of nesting of each of the different age classes of birds. The table below shows the number of fledglings raised by each of the age classes over the 4 year period. (Note that these columns total to the number of 0-1 year old birds the next year.) Use the data below to compute the average birth rates for each of the age classes b_2 and b_3. (One year old birds of this species don't nest.)

Bird Age	Year 1	Year 2	Year 3	Year 4
1-2	38	47	66	74
older	199	211	245	293

b. Write the Leslie matrix for this species of bird using the average values computed above (to **4 significant figures**). Use your Leslie matrix to estimate the population of each of the age classes for the next 3 years. (Use the last surveyed data as your starting point for this simulation.)

c. Find the eigenvalues and eigenvectors for this model, then give the limiting percent population in each of the age classes. What is the approximate annual rate of growth for this species of bird and how long would it take for the total population to double?

6. An experiment is set up with two competing species of flour beetles, where the amount of flour is maintained at a constant level. This experiment is followed for 100 days.

a. From the data, a competition model can be derived, and it has the following form:

$$\begin{aligned} x_{n+1} &= 1.075x_n - 0.00042x_n^2 - 0.00083x_ny_n, \\ y_{n+1} &= 1.038y_n - 0.00029y_n^2 - 0.00122x_ny_n, \end{aligned}$$

where n is in days and x_n and y_n represents the two species of flour beetles. Describe each of the terms in the model above. Simulate this model for 100 days assuming that

$$x_0 = 8 \qquad \text{and} \qquad y_0 = 24.$$

Show the time series graph of both populations.

b. Find all equilibria and find the eigenvalues at those equilibria. Discuss the stability of each of the equilibria and predict what will happen with the populations of these beetles over a long period of time, assuming the experimental conditions above hold. Sketch a phase portrait of the populations (x on the horizontal axis and y on the vertical axis) showing all equilibria and indicating the eigenvectors and direction of population change relative to the equilibria. Show the trajectory in the phase plane from your time series simulation above.