Spring 2007 Math 336 Name________________________

I, ______________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

Give all answers to at least **4 significant figures**. Be sure to include appropriate **units** for each of your answers.

1. An enclosed area is divided into four regions with varying habitats. One hundred tagged frogs are released into the first region. Earlier experiments found that on average the movement of frogs each day about the four regions satisfied the transition model given by

$$\begin{pmatrix} f_1(n+1) \\ f_2(n+1) \\ f_3(n+1) \\ f_4(n+1) \end{pmatrix} = \begin{pmatrix} 0.42 & 0.16 & 0.19 & 0.16 \\ 0.07 & 0.38 & 0.24 & 0.13 \\ 0.34 & 0.19 & 0.51 & 0.27 \\ 0.17 & 0.27 & 0.06 & 0.44 \end{pmatrix} \begin{pmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \\ f_4(n) \end{pmatrix}.$$

a. Give the expected distribution of the tagged frogs after 1, 2, 5, and 10 days.

b. What is the expected distribution of the frogs after a long period of time? Which of the four regions is the most suitable habitat and which is the least suitable for these frogs?

2. a. Consider a Malthusian growth model with a 3% annual growth:

$$P_{n+1} = 1.03P_n, \qquad P_0 = 100,$$

where n is in years. Determine the population after 20 years. Find the length of time for this population to double.

b. Now consider a birth only model for this population. Start with a population of 100 individuals. Each individual has a 3% chance of producing an offspring each year. Perform a Monte Carlo simulation for 20 years and give the population for each of the 20 years. Note that new individuals can give birth in subsequent years.

c. Run the simulation in Part b 1000 times and compute the average population at 10 and 20 years for these simulations. Also, compute the standard deviation for these 1000 simulations at 10 and 20 years. Compare these results to your calculations in Part a.

3. Consider the following predator-prey model

$$\begin{aligned} x_{n+1} &= 1.02x_n - 0.0002x_n^2 - 0.0008x_n y_n \\ y_{n+1} &= 0.97y_n + 0.0009x_n y_n. \end{aligned}$$

a. Which variable represents the predator and which one represents the prey in this model? Give a biological interpretation for each of the terms in the two equations above. Let $x_0 = 10$ and $y_0 = 5$ and simulate this model(time series simulation) for $n \in [0, 500]$ (*i.e.*, show graph). What are the strengths and weaknesses for this model? (Give two of each.)

b. Find all equilibria for this model. Determine the eigenvalues and associated eigenvectors at each of the equilibria, then discuss the stability of these equilibria. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral).

c. Draw a phase portrait of this model. Show at least one representative trajectory in the phase portrait and use arrows to show the directions of the solutions near each of the equilibria. What does this suggest happens over the long period of time according to this model?

4. An age-structured population of birds were surveyed over 5 years. The researchers determined the number of birds in each age class for each of the 5 years and found out how many nestlings fledged from each of the different age classes each year. These birds typically live only 4 years. Assume a typical age-structured model of the form

$$\begin{pmatrix} P_1(n+1) \\ P_2(n+1) \\ P_3(n+1) \\ P_4(n+1) \end{pmatrix} = \begin{pmatrix} 0 & b_2 & b_3 & b_4 \\ s_{12} & 0 & 0 & 0 \\ 0 & s_{23} & 0 & 0 \\ 0 & 0 & s_{34} & 0 \end{pmatrix} \begin{pmatrix} P_1(n) \\ P_2(n) \\ P_3(n) \\ P_4(n) \end{pmatrix}.$$

a. The table below shows how many birds in each age class survived to the next year (and gives the total number of birds that fledged. Use the data below to compute the average values for each of the survival parameters s_{12}, s_{23}, and s_{34}.

Bird Age	Year 1	Year 2	Year 3	Year 4	Year 5
0-1	210	275	319	371	425
1-2	75	80	105	126	145
2-3	60	62	73	99	108
3-4	38	44	46	54	68

They also collected data on the success rate of nesting of each of the different age classes of birds. The table below shows the number of fledgings raised by each of the age classes over the 5 year period. (Note that these columns total to the number of 0-1 year old birds the next year. Use the data below to compute the average birth rates for each of the age classes b_2, b_3, and b_4. (One year old birds of this species don't nest.)

Bird Age	Year 1	Year 2	Year 3	Year 4	Year 5
1-2	62	67	85	89	117
2-3	109	120	142	190	203
3-4	104	132	144	146	183

b. Write the Leslie matrix for this species of bird using the average values computed above (to **3 significant figures**). Use your Leslie matrix to estimate the population of each of the age classes for the next 5 years.

c. Find the eigenvalues and eigenvectors for this model, then give the limiting percent population in each of the age classes. What is the approximate annual rate of growth for this species of bird and how long would it take for the total population to double?

5. a. A certain yeast is placed in a chemostat with a constant supply of nutrients. Data are collected on the concentration of the yeast in this environment and are summarized in the table below:

t (hr)	Yeast
0	12.1
5	16.3
10	23.4
20	40.5
30	66.8
40	94.3
50	123.7
60	145.2
70	155.4
80	167.8
90	169.1
100	169.7

These growth conditions satisfy the assumptions for a logistic growth model of the form:

$$X_{n+1} = X_n + aX_n\left(1 - \frac{X_n}{M}\right).$$

Use the data above to find the best initial condition and parameters, X_0, a, and M to fit this model. Also, give the sum of square errors between the data and the model. Graph the data and the best fitting model.

b. A slower growing yeast is found to satisfy the logistic growth model:

$$Y_{n+1} = Y_n + bY_n\left(1 - \frac{Y_n}{N}\right),$$

with the parameters $b = 0.03475$ and $N = 112.9$ from an experiment similar to the one performed in Part a. A new experiment on competition of the two species of yeast is performed where the two species are introduced into a single chemostat and allowed to grow together. Below is a table of the results for the competition experiment.

t (hr)	X	Y
0	9.2	6.4
5	12.7	7.5
10	17.6	8.8
20	30.2	11.3
30	51.7	14.8
40	77.6	19.2
50	102.4	22.6
60	126.5	27.3
70	143.5	28.4
80	149.7	32.2
90	154.6	35.1
100	157.2	36.9

A competition model is created using the information from the single species experiments. The model has the form:

$$\begin{aligned} X_{n+1} &= X_n + aX_n\left(1 - \frac{X_n}{M}\right) - \frac{X_nY_n}{a_3}, \\ Y_{n+1} &= Y_n + bY_n\left(1 - \frac{Y_n}{N}\right) - \frac{X_nY_n}{b_3}, \end{aligned}$$

where the constants a, b, M, and N are the same as the from the single species experiments. Use the data above to find the best initial conditions and parameters, X_0, Y_0, a_3, and b_3 to fit this model. Also, give the sum of square errors between the data and the model. Graph the data and the best fitting model.

c. Find all equilibria for the competition model. Determine the eigenvalues and associated eigenvectors at each of the equilibria, then discuss the stability of these equilibria. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral).

d. Draw a phase portrait of this model. Show at least one representative trajectory in the phase portrait and use arrows to show the directions of the solutions near each of the equilibria. What does this suggest happens over the long period of time according to this model?