Partial credit may be awarded for showing correct formulae or describing how you obtained your answer on the computer.

1. An experimental plot is designed with different levels of light and divided into 3 distinct regions. One hundred tagged cockroaches (that favor darkness) are released into the experimental plot. Each hour, it is observed that in Region A, 25% of the roaches stay in Region A, 40% move to Region B, and 35% move to Region C. The roaches in Region B have 35% of the roaches staying in Region B, 15% moving to Region A, and 50% moving to Region C. The roaches in Region C have 30% of the roaches staying in Region C, 25% moving to Region A, and 45% moving to Region B.

   a. Place values on the arrows in the diagram below for this experiment. Write an appropriate transition model describing the hourly flux of cockroaches between the different plots.
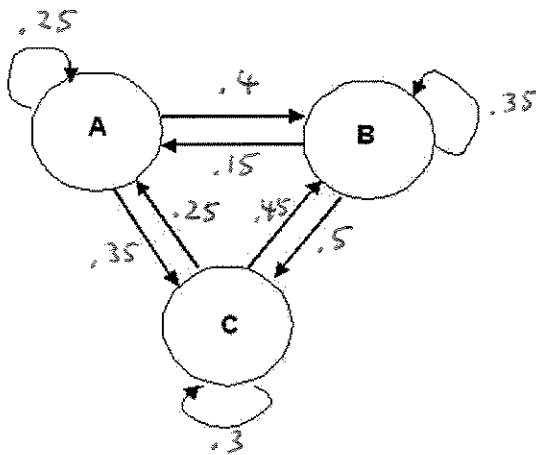
$$P_{n+1} = \begin{bmatrix} .25 & .15 & .25 \\ .4 & .35 & .45 \\ .35 & .5 & .3 \end{bmatrix} P_n$$

   b. If 100 cockroaches are released in Region A, find the expected distribution after one and two hours.

$$P_1 = T \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \\ 35 \end{pmatrix} \qquad P_2 = T\{P_1 = \begin{pmatrix} 21 \\ 39.75 \\ 39.25 \end{pmatrix}$$

   c. What is the expected distribution of the cockroaches after a long period of time? Assuming there is some experimental error (about 5%), determine which of the regions has the most light and which has the least or if this cannot be determined from this experiment. Explain.

Asymp. distrib $\begin{pmatrix} 21\% \\ 39.95\% \\ 39.04\% \end{pmatrix}$ Based on experimental error Regions B & C are equally dark. Region A is the lightest

2. a. Consider a Malthusian growth model with a 4% annual growth:

$$P_{n+1} = 1.04P_n, \qquad P_0 = 50,$$

where n is in years. Determine the population after 10 years. Find the length of time for this population to double.

$P_{10} = (1.04)^{10} \cdot 50 = 74.01$

$100 = 5(1.04)^n$

$2 = (1.04)^n$

$n = \frac{\ln(2)}{\ln(1.04)} = 17.67 \text{ yr}.$

7

b. Now consider a birth only model for this population. Start with a population of 50 individuals. Each individual has a 4% chance of producing an offspring each year. Perform a Monte Carlo simulation for 10 years and give the population for each of the 10 years. Note that new individuals can give birth in subsequent years.

$P_0 = 50$, $P_1 = 52$, $P_2 = 55$, $P_3 = 57$, $P_4 = 59$, $P_5 = 62$, $P_6 = 63$

$P_7 = 68$, $P_8 = 71$, $P_9 = 73$, $P_{10} = 75$

5

c. Run the simulation in Part b 100 [1000] times and compute the average population at 10 years for these simulations. Also, compute the standard deviation for these 100 [1000] simulations at 10 years. Compare these results to your calculations in Part a.

$\text{mean}(P_{10}) = 74.75, \quad 73.96, \quad 72.22, \quad 73.96 \quad | \quad 73.9 \quad 74.3$

$\text{std}(P_{10}) = 5.6127, \quad 5.197, \quad 5.1632, \quad 5.3009 \quad | \quad 5.92 \quad 5.7$

7

Calculations close to Part a. in the mean.

19

3. A type of mouse usually lives 3 years and raises a few broods each year. Some researchers surveyed a population of this species of mouse over a year to learn about their behavior, breeding patterns, and survival. Below are two tables on the survival and breeding from their study.

Mouse Age	Year 1	Year 2
0-1	450	710
1-2	120	95
2-3	80	40

Table 1: Survival of Mice.

Mouse Age	Year 1
0-1	355
1-2	175
2-3	180

Table 2: Reproduction success from each age group.

An age-structured population for this species of mouse has the form

$$\begin{pmatrix} P_1(n+1) \\ P_2(n+1) \\ P_3(n+1) \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & b_3 \\ s_{12} & 0 & 0 \\ 0 & s_{23} & 0 \end{pmatrix} \begin{pmatrix} P_1(n) \\ P_2(n) \\ P_3(n) \end{pmatrix}.$$

a. Use the two tables above to compute the survival parameters, s_{12} and s_{23}, and successful (living at least a year) birth rates for each of the age classes b_1, b_2, and b_3 (to **3 significant figures**). Write the Leslie matrix for this species of mouse.

$b_1 = 0.789$ $s_{12} = 0.211$

$b_2 = 1.46$ $s_{23} = 0.333$

$b_3 = 2.25$

$$L = \begin{pmatrix} 0.789 & 1.46 & 2.25 \\ 0.211 & 0 & 0 \\ 0 & 0.333 & 0 \end{pmatrix}$$

b. Use your Leslie matrix to estimate the population of each of the age classes for Years 3 and 5. Give the total population for each of the first 5 years.

$$P(3) = L \begin{pmatrix} 710 \\ 95 \\ 40 \end{pmatrix} = \begin{pmatrix} 788.9 \\ 149.8 \\ 31.635 \end{pmatrix} \qquad P(5) = \begin{pmatrix} 1075 \\ 192.5 \\ 55.4 \end{pmatrix}$$

$Tot(3) = 970.3$ $Tot(5) = 1323$

$Tot(4) = 1,128.7$

c. Find the eigenvalues and eigenvectors for this model, then give the limiting percent population in each of the age classes. What is the approximate annual rate of growth for this species of mouse and how long would it take for the total population to double?

$\lambda_1 = 1.1684$ $\lambda_{2,3} = -0.18977 \pm 0.3151i$ $x_2 = \begin{pmatrix} 0.7909 \\ -0.2340 \mp 0.3287i \\ -0.1922 \pm 0.3630i \end{pmatrix}$

$x_1 = \begin{pmatrix} 0.8117 \\ 0.1466 \\ 0.0418 \end{pmatrix}$ Annual growth ~ 17%

← limiting distribution

$t_d \approx 4.454$ yrs

8

7

15

30

4. A study of two organisms living together in a particular environment show that they satisfy the following model:

$$X_{n+1} = 1.1X_n - 0.02X_n^2 + 0.01X_nY_n,$$
$$Y_{n+1} = 1.2Y_n - 0.01Y_n^2 + 0.01X_nY_n,$$

where species X_n and Y_n are measured in thousands of organisms.

a. Describe each of the terms in the above model and how they reflect the interaction of the species with their own kind or with the other species. What is the relationship between these species?

First, term for each equation is a Malthusian growth term

Second term is intraspecies competition

Last term is mutualism with each species helping other

This is a symbiotic relationship.

b. Find all equilibria for this model. Determine the eigenvalues and associated eigenvectors at each of the equilibria, then discuss the stability of these equilibria. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral). Draw a phase portrait of this model and show at least one representative trajectory. Use arrows to show the directions of the solutions near each of the equilibria.
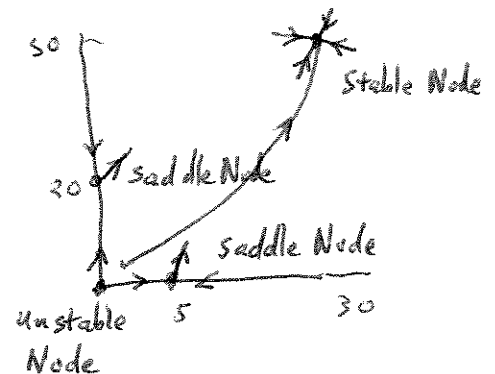
Equil. $(X_e, Y_e) = (0,0), (5,0), (0,20), (30,50)$

At $(0,0)$, $\lambda_1 = 1.1, [1,0], \lambda_2 = 1.2, [0,1]$ ✓

At $(5,0)$, $\lambda_1 = 0.9, [1,0], \lambda_2 = 1.25, [0.143,1]$

At $(0,20)$, $\lambda_1 = 0.8, [1,1], \lambda_2 = 1.3, [1,0.4]$

At $(30,50)$, $\lambda_1 = 0.84, [-0.581,-0.853]$
$\lambda_2 = 0.059, [-0.661, 0.75]$

c. If this environment begins with 1000 individuals of each species ($X_0 = 1$ and $Y_0 = 1$), then what is the eventual outcome according to this model? How would you characterize the eventual outcome in the environment if the initial colonization consists of only 1000 individuals of species X_n, so $X_0 = 1$ and $Y_0 = 0$?

Starting $(1,1) \rightarrow (30,50)$ eventually (mutual cooperative equil)

Starting $(1,0) \rightarrow (5,0)$ eventually (carrying capacity for X)