1. Solve the following initial value problems using the method of Laplace transforms. (Thus, you must show the problem in the transform space and the solution in t.)

a. $\ddot{y} - 4\dot{y} = 20 - \cos(2t), \qquad y(0) = 2, \quad \dot{y}(0) = -1$

b. $\ddot{y} + 6\dot{y} + 9y = 18t^2 + 6e^{-3t}, \qquad y(0) = 3, \quad \dot{y}(0) = -2$

c. $\ddot{y} + 4\dot{y} + 8y = \begin{cases} 1 & 0 \le t < 1 \\ 1 - t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}, \qquad y(0) = -1, \quad \dot{y}(0) = 2$

2. A bridge can be considered to be a harmonic oscillator. When someone is walking across the bridge, their steps impart an impulsive force. Below we examine two cases of impulsive force applied to the harmonic oscillator.

a. Solve the initial value problem:

$$\frac{d^2y}{dt^2} + y = \sum_{j=0}^{\infty} \delta(t - j\pi), \qquad y(0) = 0 \qquad y'(0) = 0.$$

b. Solve this initial value problem:

$$\frac{d^2y}{dt^2} + y = \sum_{j=0}^{\infty} \delta(t - 2j\pi), \qquad y(0) = 0 \qquad y'(0) = 0.$$

c. Use the information above to explain why soldiers are instructed to break cadence when marching across a bridge. (A historical note: In the 17^{th} century, a number of British soldiers died when they marched across a bridge and set up resonance so that the bridge collapsed.)

3. a. Show that, if $\mathcal{L}\{f\} = F(s)$, then

$$\mathcal{L}\{t\, f(t)\} = -\frac{dF(s)}{ds}.$$

b. From the result above, derive the formula for $\mathcal{L}\{t \sin(\omega t)\}$.

4. a. Let f be a periodic function with period T (with $T > 0$), then $f(t+T) = f(t)$ for all $t \geq 0$. Show that

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt.$$

b. Use the result above to find the Laplace transform of the saw tooth function (see figure below), which is defined by

$$f(t) = t \quad \text{for} \quad 0 \leq t < 1; \qquad f(t+1) = f(t).$$