1. (8pts) The companion matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & -18 & -10 & -2 \end{pmatrix},$$

has the characteristic polynomial:

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda+1)^2(\lambda^2+9) = 0.$$

The eigenvalue, $\lambda_1 = -1$ has algebraic multiplicity 2 and geometric multiplicity 1. It has an eigenvector $\xi_1 = [1, -1, 1, -1]^T$. We find the eigenvector ξ_2 in the higher null space satisfies:

$$(A+I)\xi_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -9 & -18 & -10 & -1 \end{pmatrix} \xi_2 = \xi_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{so} \quad \xi_2 = \begin{pmatrix} 1+s \\ 0-s \\ -1+s \\ 2-s \end{pmatrix} \quad \text{for any } s.$$

The eigenvalue, $\lambda_3 = 3i$ is complex (algebraic and geometric multiplicity 1). It has a complex eigenvector $\xi_1 = [1, 3i, -9, -27i]^T$. With the information from the eigenvalue, $\lambda_1 = -1$, and taking the real and imaginary parts of the eigenvector for $\lambda_3 = 3i$, we obtain a transformation matrix:

$$P = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 1 & -1 & -9 & 0 \\ -1 & 2 & 0 & -27 \end{pmatrix} \qquad \text{with} \qquad P^{-1} = \begin{pmatrix} \frac{9}{50} & -\frac{18}{25} & \frac{1}{50} & -\frac{2}{25} \\ \frac{9}{10} & \frac{9}{10} & \frac{1}{10} & \frac{1}{10} \\ -\frac{2}{25} & -\frac{9}{50} & -\frac{3}{25} & -\frac{1}{50} \\ \frac{3}{50} & \frac{7}{75} & \frac{1}{150} & -\frac{2}{75} \end{pmatrix}.$$

It follows that the real Jordan canonical form given by:

$$P^{-1}AP = J = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -3 & 0 \end{pmatrix}.$$

The system of linear ODEs

$$\dot{\mathbf{y}} = J\mathbf{y},$$

has the real fundamental solution,

$$\Psi(t) = \begin{pmatrix} e^{-t} & te^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & \cos(3t) & \sin(3t) \\ 0 & 0 & -\sin(3t) & \cos(3t) \end{pmatrix}.$$

2. (8pts) The companion matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 25 & 15 & -14 & -18 & -7 \end{pmatrix},$$

has the characteristic polynomial:

$$\lambda^5 + 7\lambda^4 + 18\lambda^3 + 14\lambda^2 - 15\lambda - 25 = (\lambda - 1)(\lambda^2 + 4\lambda + 5)^2 = 0.$$

Since this is a companion matrix, the eigenvalue, $\lambda_1 = 1$ (algebraic and geometric multiplicity 1) has an eigenvector $\xi_1 = [1, 1, 1, 1, 1]^T$. The other eigenvalues, $\lambda = -2 \pm i$, are repeated complex eigenvalues. The eigenvalue, $\lambda_2 = -2 + i$ (algebraic multiplicity 2 and geometric multiplicity 1), has complex eigenvector is $\xi_2 = [1, -2 + i, 3 - 4i, -2 + 11i, -7 - 24i]^T$. We find the eigenvector ξ_3 in the higher null space satisfies:

$$(A - (-2 + i)I)\xi_2 = \begin{pmatrix} 2-i & 1 & 0 & 0 & 0 \\ 0 & 2-i & 1 & 0 & 0 \\ 0 & 0 & 2-i & 1 & 0 \\ 0 & 0 & 0 & 2-i & 1 \\ 25 & 15 & -14 & -18 & -5-i \end{pmatrix} \xi_3 = \xi_2 = \begin{pmatrix} 1 \\ -2+i \\ 3-4i \\ -2+11i \\ -7-24i \end{pmatrix},$$

so

$$\xi_3 = \begin{pmatrix} 0 \\ 1 \\ -4+2i \\ 9-12i \\ -8+44i \end{pmatrix} + s \begin{pmatrix} 1 \\ -2+i \\ 3-4i \\ -2+11i \\ -7-24i \end{pmatrix} \qquad \text{for any } s.$$

With the information from the eigenvalue, $\lambda_1 = 1$, and taking the real and imaginary parts of the eigenvector for $\lambda_2 = -2 + i$ and the real and imaginary parts of the eigenvector, ξ_3, in the higher null space, we obtain a transformation matrix:

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 & 0 \\ 1 & 3 & -4 & -4 & 2 \\ 1 & -2 & 11 & 9 & -12 \\ 1 & -7 & -24 & -8 & 44 \end{pmatrix}$$

It follows that the real Jordan canonical form given by:

$$P^{-1}AP = J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 & -2 \end{pmatrix}.$$

The system of linear ODEs

$$\dot{\mathbf{y}} = J\mathbf{y},$$

has the real fundamental solution,

$$\Psi(t) = \begin{pmatrix} e^t & 0 & 0 & 0 & 0 \\ 0 & e^{-2t}\cos(t) & e^{-2t}\sin(t) & te^{-2t}\cos(t) & te^{-2t}\sin(t) \\ 0 & -e^{-2t}\sin(t) & e^{-2t}\cos(t) & -te^{-2t}\sin(t) & te^{-2t}\cos(t) \\ 0 & 0 & 0 & e^{-2t}\cos(t) & e^{-2t}\sin(t) \\ 0 & 0 & 0 & -e^{-2t}\sin(t) & e^{-2t}\cos(t) \end{pmatrix}.$$