1. (2pts) The fourth order scalar ODE given by:

$$y'''' - 16\,y = 0,$$

with $y_1(t) = y(t)$, $y_2 = \dot{y}_1$, $y_3 = \dot{y}_2$, and $y_4 = \dot{y}_3$ satisfies $\dot{y}_4 = 16y_1$. It is easy to see that a first order linear system can be written:

$$\dot{\mathbf{y}} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = A\mathbf{y}. \tag{1}$$

2. (4pts) From Eqn. (1), the eigenvalues satisfy the characteristic equation:

$$\det|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 16 & 0 & 0 & -\lambda \end{vmatrix} = 0.$$

Expanding the determinant by the first column gives:

$$\det|A - \lambda I| = -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 16 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \lambda^4 - 16 = 0.$$

It follows that the eigenvalues are $\lambda = 2,\ -2,\ 2i,\ -2i$. This form of matrix can readily be shown to have eigenvectors of the form, $\xi_i = [1, \lambda, \lambda^2, \lambda^3]^T$, so the associated eigenvectors are:

$$\text{For} \quad \lambda_1 = 2, \quad \xi_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix}, \qquad \text{for} \quad \lambda_2 = -2, \quad \xi_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix},$$

$$\text{for} \quad \lambda_3 = 2i, \quad \xi_3 = \begin{pmatrix} 1 \\ 2i \\ -4 \\ -8i \end{pmatrix}, \qquad \text{for} \quad \lambda_4 = -2i, \quad \xi_4 = \begin{pmatrix} 1 \\ -2i \\ -4 \\ 8i \end{pmatrix}.$$

3. (4pts) The complex solution of Eqn. (1) is readily written from the e.v.s and e.f.s and has the form:

$$\begin{aligned} \mathbf{y}(t) &= c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2i \\ -4 \\ -8i \end{pmatrix} e^{2it} + c_4 \begin{pmatrix} 1 \\ -2i \\ -4 \\ 8i \end{pmatrix} e^{-2it} \\ &= c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 1 \\ 2i \\ -4 \\ -8i \end{pmatrix} \Big(\cos(2t) + i\sin(2t)\Big) + c_4 \begin{pmatrix} 1 \\ -2i \\ -4 \\ 8i \end{pmatrix} \Big(\cos(2t) - i\sin(2t)\Big). \end{aligned}$$

The real solution of of Eqn. (1) is readily written using the real and imaginary parts of the complex solution, so

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 4 \\ -8 \end{pmatrix} e^{-2t} + d_3 \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \\ -4\cos(2t) \\ 8\sin(2t) \end{pmatrix} + d_4 \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \\ -4\sin(2t) \\ -8\cos(2t) \end{pmatrix}.$$

4. (6pts) A Hermite differential equation satisfies:

$$y'' - 2xy' + 10y = 0.$$

Assume a power series of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}.$$

Substituting these into the Hermite ODE gives:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^n + 10\sum_{n=0}^{\infty} a_n x^n = 0.$$

By shifting the dummy index of the first term and noting the second term can start at $n = 0$, we have

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty}(2n-10)a_n x^n = \sum_{n=0}^{\infty}\left[a_{n+2}(n+2)(n+1) - (2n-10)a_n\right]x^n = 0.$$

The coefficient of x^n gives the *recurrence relation*:

$$a_{n+2}(n+2)(n+1) = (2n-10)a_n \qquad \text{or} \qquad a_{n+2} = \frac{2n-10}{(n+2)(n+1)}a_n.$$

Since this is a second order ODE, there are the two arbitrary constants, $y(0) = a_0$ and $y'(0) = a_1$. From the recurrence formula, the other coefficients are obtained with the table below showing the coefficients to powers of $n = 8$.

$$\begin{array}{ll} a_2 = -5a_0 & a_3 = \frac{-8a_1}{3\cdot 2} = -\frac{4}{3}a_1 \\ a_4 = \frac{-6a_2}{4\cdot 3} = \frac{5}{2}a_0 & a_5 = \frac{-4a_3}{5\cdot 4} = \frac{4}{15}a_1 \\ a_6 = \frac{-2a_4}{6\cdot 5} = -\frac{a_0}{6} & a_7 = 0 = a_9 = a_{11} = \dots = a_{2n+1}, \quad n \geq 3 \\ a_8 = \frac{2a_6}{8\cdot 7} = -\frac{1}{168}a_0 & \end{array}$$

It follows that the two linearly independent solutions are:

$$\begin{aligned} y_1(x) &= a_0\left(1 - 5x^2 + \frac{5}{2}x^4 - \frac{1}{6}x^6 - \frac{1}{168}x^8 + \dots\right) \\ y_2(x) &= a_1\left(x - \frac{4}{3}x^3 + \frac{4}{15}x^5\right). \end{aligned}$$

It is clear that y_2 is a polynomial of order 5.