1. (20pts) a. For the differential equation

$$\dot{v} + \tilde{k}\cos(\omega t)v = -g,$$

we use the scaling arguments $v = \alpha\tilde{v}$ and $\tau = \beta t$. It follows that

$$\alpha\beta\frac{d\tilde{v}}{d\tau} + \alpha\tilde{k}\cos\left(\frac{\omega}{\beta}\tau\right)\tilde{v} = -g$$

or

$$\frac{d\tilde{v}}{d\tau} + \frac{\tilde{k}}{\beta}\cos\left(\frac{\omega}{\beta}\tau\right)\tilde{v} = -\frac{g}{\alpha\beta}.$$

We take $\frac{g}{\alpha\beta} = 1$, $\omega = \beta$, $\gamma = \frac{\tilde{k}}{\beta} = \frac{\tilde{k}}{\omega}$, so $\alpha = \frac{g}{\omega}$, resulting in the desired scaled equation:

$$\frac{d\tilde{v}}{d\tau} + \gamma\cos(\tau)\tilde{v} = -1.$$

b. The above ODE is linear with an integrating factor:

$$\mu(\tau) = e^{\int_0^\tau \gamma\cos(s)ds} = e^{\gamma\sin(\tau)}.$$

With the initial condition $\tilde{v}(0) = \tilde{v}_0$, it follows that the solution to the ODE is

$$\tilde{v}(\tau) = e^{-\gamma\sin(\tau)}\left(\tilde{v}_0 - \int_0^\tau e^{\gamma\sin(s)}ds\right).$$

As $e^{-\gamma\sin(\tau)}$ oscillates for all time, the term with $\tilde{v}_0$ is always present in the solution, so matters. However, since $e^{\gamma\sin(s)} \geq e^{-|\gamma|}$, it follows that

$$\int_0^\tau e^{\gamma\sin(s)}ds \geq e^{-|\gamma|}\int_0^\tau ds = \tau e^{-|\gamma|},$$

which is unbounded for large τ, making the $\tilde{v}_0$ term insignificant.

To examine for a 2π periodic solution, we consider

$$\begin{aligned}\tilde{v}(\tau + 2\pi) &= \tilde{v}_0 e^{-\gamma\sin(\tau)} - e^{-\gamma\sin(\tau)}\left(\int_0^{2\pi} e^{\gamma\sin(s)}ds + \int_{2\pi}^{\tau+2\pi} e^{\gamma\sin(s)}ds\right)\\ &= \tilde{v}(\tau) - e^{-\gamma\sin(\tau)}\int_0^{2\pi} e^{\gamma\sin(s)}ds.\end{aligned}$$

Thus, for $\tilde{v}(\tau)$ to be 2π periodic, then

$$\int_0^{2\pi} e^{\gamma\sin(s)}ds = 0,$$

which is impossible as $e^{\gamma\sin(s)} > 0$.

c. For γ small, the Maclaurin series expansion of $e^{\gamma\sin(s)}$ is

$$e^{\gamma\sin(s)} = 1 + \gamma\sin(s) + \tfrac{1}{2}\gamma^2\sin^2(s) + \tfrac{1}{6}\gamma^3\sin^3(s) + \mathcal{O}\left(\gamma^4\right).$$

This is readily integrated with $\sin^2(s) = \frac{1}{2}(1-\cos(2s))$ and $\sin^3(s) = (1-\cos^2(s))\sin(s)$:

$$\begin{aligned}
\int_0^\tau e^{\gamma\sin(s)}ds &= \left(s - \gamma\cos(s) + \frac{\gamma^2}{2}\left(\frac{s}{2} - \frac{\sin(2s)}{4}\right) + \frac{\gamma^3}{6}\left(-\cos(s) + \frac{\cos^3(s)}{3}\right) + \mathcal{O}\left(\gamma^4\right)\right)\Bigg|_0^\tau \\
&= \tau - \gamma\left(\cos(\tau) - 1\right) + \frac{\gamma^2}{4}\left(\tau - \frac{\sin(2\tau)}{2}\right) + \frac{\gamma^3}{6}\left(\frac{2}{3} - \cos(\tau) + \frac{\cos^3(\tau)}{3}\right) \\
&\quad + \mathcal{O}\left(\gamma^4\right).
\end{aligned}$$

The terms with τ $(\tau + \frac{\gamma^2}{4}\tau)$ are non-periodic.

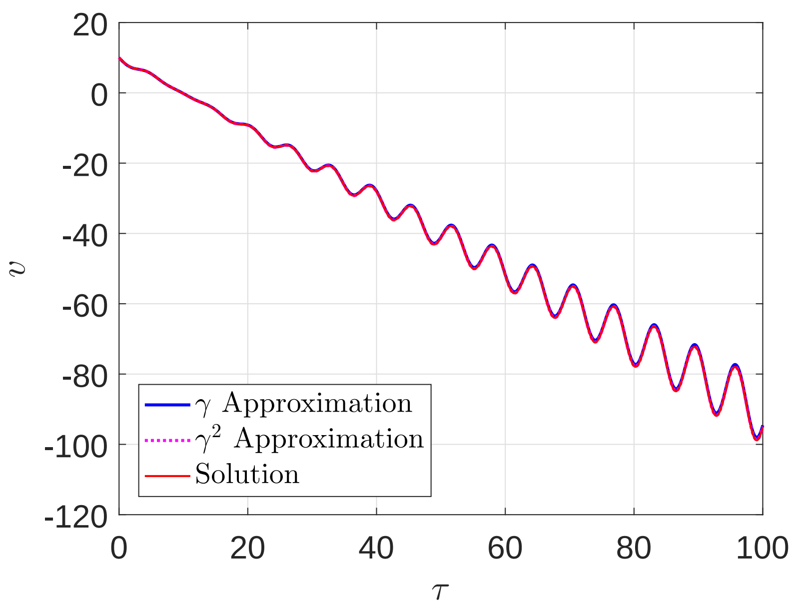
d. We use the Taylor series expansion:

$$e^{-\gamma\sin(\tau)} = 1 - \gamma\sin(\tau) + \frac{\gamma^2}{2}\sin^2(\tau) + \mathcal{O}\left(\gamma^3\right),$$

with the solution above and the series expansion in Part c to obtain our expansion. Thus, we have (with $\sin(2\tau) = 2\sin(\tau)\cos(\tau)$):

$$\begin{aligned}
\tilde{v}(\tau) &= e^{-\gamma\sin(\tau)}\left(\tilde{v}_0 - \int_0^\tau e^{\gamma\sin(s)}ds\right), \\
&= \left(1 - \gamma\sin(\tau) + \frac{\gamma^2}{2}\sin^2(\tau) + \mathcal{O}\left(\gamma^3\right)\right)\cdot \\
&\quad \left(\tilde{v}_0 - \tau + \gamma\left(\cos(\tau) - 1\right) - \frac{\gamma^2}{4}\left(\tau - \sin(\tau)\cos(\tau)\right) + \mathcal{O}\left(\gamma^3\right)\right), \\
&= \left(\tilde{v}_0 - \tau\right) + \gamma\left(\cos(\tau) - 1 + \tau\sin(\tau) - \tilde{v}_0\sin(\tau)\right) \\
&\quad + \frac{\gamma^2}{2}\left(\tilde{v}_0\sin^2(\tau) - \tau\sin^2(\tau) - \frac{3\sin(\tau)\cos(\tau)}{2} - \frac{\tau}{2} + 2\sin(\tau)\right) + \mathcal{O}\left(\gamma^3\right).
\end{aligned}$$

e. Let $\gamma = 0.1$ and $\tilde{v}(0) = 10$. Below is the graph showing the 1^{st} order approximation in γ, the 2^{nd} order approximation in γ, and the "exact" solution via MatLab's *ode23* solver. We observe that the 1^{st} order approximation in γ varies a small amount from the actual solution for larger t, while the 2^{nd} order approximation, γ^2, is almost indistinguishable from the exact solution.



```matlab
v0 = 10; ep = 0.1;
tt = linspace(0,100,1000);
vv1 = -tt+v0+(-1+cos(tt)+sin(tt).*tt-sin(tt)*v0)*ep;
vv2 = vv1+((1/4)*sin(tt).*cos(tt)-(1/4)*tt-sin(tt).*(-1+cos(tt))...
    -(1/2)*(sin(tt).^2).*tt+(1/2)*sin(tt).^2*v0)*ep^2;
[t1,z1] = ode23(@hw2_1d,tt,v0);
plot(tt,vv1,'b-','LineWidth',1.5);
hold on
plot(tt,vv2,'m:','LineWidth',1.5);
plot(t1,z1,'r-','LineWidth',1.0);
grid;
h = legend('$\gamma$ Approximation', '$\gamma^2$ Approximation',...
    'Solution','Location','southwest');
set(h,'Interpreter','latex')
xlim([0,100]);
ylim([-120,20]);
% Set up fonts and labels for the Graph
fontlabs = 'Times New Roman';
xlabel('$\tau$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
ylabel('$v$','FontSize',16,'FontName',fontlabs, ...
    'interpreter','latex');
mytitle = '';
title(mytitle,'FontSize',16,'FontName', ...
    'Times New Roman','interpreter','latex');
set(gca,'FontSize',16);
print -depsc hw2_1d.eps    % Create EPS file (Figure)
```

```matlab
function zp = hw2_1d(T,z)
%Perturbation ODE
ep = 0.1;
zp = -ep*cos(T)*z -1;
end
```

2. (6pts) Consider the 2^{nd} order linear ODE (Legendre's DE when $\gamma = n(n+1)$):

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \gamma y = 0. \tag{1}$$

We make the transformation $x = \cos(\theta)$, so

$$\frac{d}{d\theta} = \frac{d}{dx}\left(\frac{dx}{d\theta}\right) = -\sin(\theta)\frac{d}{dx}.$$

Multiply (1) by $\sin(\theta)$ and substitute for x gives:

$$\sin(\theta)\frac{d}{dx}\left((1-\cos^2(\theta))\frac{dy}{dx}\right) + \sin(\theta)\gamma y = 0,$$

which is equivalent to

$$\sin(\theta)\frac{d}{dx}\left(\sin^2(\theta))\frac{dy}{dx}\right)+\sin(\theta)\gamma y=0.$$

This can be rewritten:

$$-\sin(\theta)\frac{d}{dx}\left(\sin(\theta)\left(-\sin(\theta)\frac{dy}{dx}\right)\right)+\sin(\theta)\gamma y=0.$$

Changing differential operators from the transformation above gives:

$$\frac{d}{d\theta}\left(\sin(\theta)\frac{dy}{d\theta}\right)+\sin(\theta)\gamma y=0,$$

(in Sturm-Liouville form), which can be expanded to

$$\frac{d^2y}{d\theta^2}+\cot(\theta)\frac{dy}{d\theta}+\gamma y=0.$$

This problem is important in solving spherical PDEs in the ϕ-direction.

3. (10pts) a. Consider

$$\frac{dy}{dt}+\alpha\cos(\omega t)y=\beta\sin(\omega t),\qquad y(0)=y_0,$$

where $\omega/\alpha=200$. We let $y=A\tilde{y}$, $\tau=\omega t$, and $\tilde{y}(0)=\tilde{y}_0$. Then the rescaled ODE becomes:

$$A\omega\frac{d\tilde{y}}{d\tau}+A\alpha\cos(\tau)\tilde{y}=\beta\sin(\tau)$$

or

$$\frac{d\tilde{y}}{d\tau}+\frac{\alpha}{\omega}\cos(\tau)\tilde{y}=\frac{\beta}{A\omega}\sin(\tau).$$

For $A=\frac{\beta}{\alpha}$, this equation becomes

$$\frac{d\tilde{y}}{d\tau}+\frac{\alpha}{\omega}\cos(\tau)\tilde{y}=\frac{\alpha}{\omega}\sin(\tau),$$

so we let $\epsilon=\frac{\alpha}{\omega}=\frac{1}{200}$ (which is small), giving

$$\frac{d\tilde{y}}{d\tau}+\epsilon\cos(\tau)\tilde{y}=\epsilon\sin(\tau),\qquad \tilde{y}(0)=\tilde{y}_0.$$

b. The above linear ODE has the integrating factor:

$$\mu(\tau)=e^{\int_0^\tau \epsilon\cos(s)ds}=e^{\epsilon\sin(\tau)}.$$

It follows that the solution satisfies:

$$\tilde{y}(\tau)=e^{-\epsilon\sin(\tau)}\left(\tilde{y}_0+\int_0^\tau \epsilon\sin(s)e^{\epsilon\sin(s)}ds\right).$$

We expand the exponentials to $\mathcal{O}(\epsilon^2)$, *e.g.*,

$$e^{-\epsilon \sin(\tau)} = 1 - \epsilon \sin(\tau) + \mathcal{O}(\epsilon^2).$$

The variation of constants formula can be written:

$$\begin{aligned}
\tilde{y}(\tau) &= \tilde{y}_0 \left(1 - \epsilon \sin(\tau) + \mathcal{O}(\epsilon^2)\right) \\
&\quad + \left(1 - \epsilon \sin(\tau) + \mathcal{O}(\epsilon^2)\right) \epsilon \int_0^\tau \sin(s) \left(1 + \epsilon \sin(s) + \mathcal{O}(\epsilon^2)\right) ds \\
&= \tilde{y}_0 + \epsilon \left[-\tilde{y}_0 \sin(\tau) + \int_0^\tau \sin(s) ds \right] + \mathcal{O}(\epsilon^2),
\end{aligned}$$

so

$$\tilde{y}(\tau) = \tilde{y}_0 + \epsilon \left[-\tilde{y}_0 \sin(\tau) - \cos(\tau) + 1 \right] + \mathcal{O}(\epsilon^2).$$

It follows that for

$$\tilde{y}(\tau) = \tilde{y}_0 + \epsilon \tilde{y}_1(\tau) + \mathcal{O}(\epsilon^2),$$

we have $\tilde{y}_1(\tau) = 1 - \cos(\tau) - \tilde{y}_0 \sin(\tau)$, which is an oscillatory solution (no limit), causing the overall solution to have small amplitude oscillations ($\mathcal{O}(\epsilon)$) about the constant solution, $\tilde{y}_0$ for small ϵ.