Fall 2021 Math 537

Take-Home Exam 1 Due Tues. 10/26/21

1. (15pts) Solve the differential equation

$$\varepsilon\frac{dy}{dt} + y = f(t/\varepsilon), \quad y(0) = y_0,$$

where $f(\xi)$ is periodic with period T, *i.e.*, $f(\xi + T) = f(\xi)$, and on $[0, T]$, $f(\xi)$ is defined to be

$$f(\xi) = \begin{cases} 1 & 0 \le \xi < T/2, \\ 0 & T/2 \le \xi < T. \end{cases}$$

Take $t \ge 0$ and be sure your solution extends at least to $3T$ (preferably for all t). Is your solution $y(t)$ periodic? Explain. (Hint: Scaling is your friend. Also, as f is a periodic step function, you might find techniques from Laplace Transforms simplify the solution. Laplace techniques are available at
https://jmahaffy.sdsu.edu/courses/f15/math337/beamer/LaplaceB-04.pdf.)
Bonus: Produce a graph for your solution with $\varepsilon = 0.1$, $T = 1$, $y_0 = 0$, and $t \in [0, 0.5]$.

2. (10pts) Show that if $||A|| < 1$, then one has that $(I - A)^{-1}$ exists and

$$\left|\left|(I - A)^{-1}\right|\right| \le \frac{1}{1 - ||A||}.$$

Hint: To pull this off, look at the limit as $N \to \infty$ of the matrices $\tilde{A}_N$ where

$$\tilde{A}_N = (I - A)\sum_{j=0}^{N} A^j - I.$$

In particular, figure out

$$\lim_{N\to\infty} \left|\left|\tilde{A}_N\right|\right|.$$

You should now find $(I - A)^{-1}$ as an infinite series. Use properties of the norm to get the bound.

3. (20pts) a. Consider the 1^{st} order system of differential equations given by:

$$\dot{\mathbf{x}} = \begin{pmatrix} -3 & -1 \\ 3-\alpha & -3 \end{pmatrix}\mathbf{x},$$

where α is a real parameter. Find the characteristic equation and eigenvalues in terms of α.

b. There are **two** critical values of α ($\alpha_1 < \alpha_2$), where the qualitative nature of the phase portrait changes. (For example, unstable node to unstable spiral or saddle node to stable node.) Determine values of α where the type of node changes for the origin. Characterize the values of the eigenvalues for $\alpha < \alpha_1$, $\alpha = \alpha_1$, $\alpha \in (\alpha_1, \alpha_2)$, $\alpha = \alpha_2$, and $\alpha > \alpha_2$. State clearly the type of behavior (such as **STABLE NODE**) for each of these values of α.

c. Sketch a typical phase portrait for each of the **5** regions above, $\alpha < \alpha_1$, $\alpha = \alpha_1$, $\alpha \in (\alpha_1, \alpha_2)$, $\alpha = \alpha_2$, and $\alpha > \alpha_2$. (You may take one value in the intervals above and use either Maple or MatLab to draw the phase portrait.) Be sure to include representative trajectories in each phase portrait.

4. (25pts) Consider the linear system of ODEs given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

For each of the following matrices A, find a **transition matrix** P that transforms A into the **real Jordan canonical form**, J. Write both P and J. Furthermore, give a **fundamental solution**, $\mathbf{\Psi}(t) = e^{Jt}$.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -3 & 0 \\ -9 & 0 & -6 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 16 & -8 & 4 \end{pmatrix}.$$

Write some details on how you obtain the **eigenvalues** and **eigenvectors** (other than just letting **Maple** do the work, understanding it can help). Provide some details on how the **transition matrix** is formed.

5. (30pts) a. Consider the nonhomogeneous system of linear ODEs:

$$\dot{\mathbf{x}} = \begin{pmatrix} -\alpha & 1 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-\gamma t} \\ 0 \\ 0 \\ \sin(\omega t) \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -2 \end{pmatrix},$$

where $\alpha, \beta, \gamma, \omega > 0$. Find the solution for this initial value problem. In addition to the generic cases, $\alpha \neq \gamma$ and $\beta \neq \omega$, solve the special cases $\alpha = \gamma$ and $\beta = \omega$. Are any solutions unbounded?

b. Consider the non-constant, nonhomogeneous system of linear ODEs with $t > 0$:

$$\dot{\mathbf{y}} = \begin{pmatrix} 0 & 1 \\ \frac{3}{t^2} & \frac{1}{t} \end{pmatrix} \mathbf{y} + \begin{pmatrix} -16t^2 \\ 8t \end{pmatrix}, \qquad \mathbf{y}(1) = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

Find the solution for this initial value problem.