1. a. This is a linear differential equation,so it can be written

$$\frac{dy}{dt} + (0.2t - 2)y = 0, \qquad \text{with} \qquad \mu(t) = e^{\int (0.2t-2)dt} = e^{0.1t^2-2t},$$

where $\mu(t)$ is the integrating factor. It follows:

$$\frac{d}{dt}\left(e^{0.1t^2-2t}y\right) = 0 \qquad \text{or} \qquad e^{0.1t^2-2t}y(t) = C$$

It follows that $y(t) = Ce^{2\,t-0.1\,t^2}$. The initial condition $y(0) = 10 = C$. Hence, the solution is

$$y(t) = 10\,e^{2\,t-0.1\,t^2}.$$

b. This is a time varying differential equation. It can be written

$$y(t) = \int \left(2 - \frac{4}{t}\right) dt = 2\,t - 4\,\ln(t) + C.$$

The initial condition $y(1) = 5 = 2+C$, which implies $C = 3$. Hence, the solution is $y(t) = 2\,t - 4\,\ln(t) + 3$.

c. This is a separable differential equation. It can be written

$$\int 2y\,dy = \int 3t^2\,dt \qquad \text{or} \qquad y^2(t) = t^3 + C.$$

It follows that $y(t) = \pm\sqrt{t^3+C}$. The initial condition $y(0) = 4 = \sqrt{C}$, which implies $C = 16$. Hence, the solution is

$$y(t) = \sqrt{t^3+16}.$$

d. This is the logistic growth differential equation, which can be written

$$\frac{dy}{dt} = 0.02y\left(1 - \frac{y}{40}\right) \qquad \text{or} \qquad \frac{dy}{dt} - 0.02y = -0.0005y^2,$$

which is a Bernoulli's equation. Make the substitution $u = y^{1-2} = y^{-1}$, so $\frac{du}{dt} = -y^{-2}\frac{dy}{dt}$. Multiply the equation above by $-y^{-2}$, and

$$-y^{-2}\frac{dy}{dt} + 0.02y^{-1} = 0.0005 \qquad \text{or} \qquad \frac{du}{dt} + 0.02u = 0.0005,$$

which is a linear equation with integrating factor $\mu(t) = e^{0.02t}$. Thus,

$$\frac{d}{dt}\left(e^{0.02t}u\right) = 0.0005e^{0.02t} \qquad \text{or} \qquad e^{0.02t}u(t) = 0.025e^{0.02t} + C.$$

Hence, with the initial condition

$$\frac{1}{y(t)} = u(t) = 0.025 + Ce^{-0.02t} \quad \text{or} \quad 0.1 = 0.025 + C, \quad \text{so} \quad C = 0.075.$$

It follows that

$$y(t) = \frac{1}{0.025 + 0.075e^{-0.02t}} = \frac{40}{1 + 3e^{-0.02t}}.$$

e. Rewrite the equation as

$$3y - 6t + (3t + 4y)\frac{dy}{dt} = 0.$$

Since $\frac{\partial M(t,y)}{\partial y} = 3 = \frac{\partial N(t,y)}{\partial t}$, this equation is exact. Integrating we see

$$\int (3y - 6t)dt = 3ty - 3t^2 + h(y) \quad \text{and} \quad \int (3t + 4y)dy = 3ty + 2y^2 + k(t).$$

It is clear that the potential function is

$$\phi(t, y) = 3ty - 3t^2 + 2y^2 = C.$$

With the initial condition $y(0) = 4$, the solution becomes

$$\phi(t, y) = 3ty - 3t^2 + 2y^2 = 32.$$

f. This linear DE equation can be rewritten

$$\frac{dy}{dt} - \frac{2y}{t} = 4t^2 \sin(4t), \qquad \text{so} \qquad \mu(t) = e^{-\int 2dt/t} = \frac{1}{t^2}.$$

Thus,

$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = 4\sin(4t) \qquad \text{or} \qquad \frac{y(t)}{t^2} = -\cos(4t) + C.$$

It follows that

$$y(t) = Ct^2 - t^2\cos(4t), \quad \text{so} \quad 2 = C - \cos(4) \quad \text{or} \quad C = 2 + \cos(4).$$

Hence, the solution is

$$y(t) = (2 + \cos(4))t^2 - t^2\cos(4t).$$

g. This is a linear and separable differential equation. We solve this time using separable techniques. The equation can be written

$$\int \frac{dy}{y} = \int \frac{2\,t\,dt}{t^2 + 1}.$$

The right integral uses the substitution $u = t^2 + 1$, so $du = 2\,t\,dt$. Hence,

$$\begin{aligned} \ln|y(t)| &= \int \frac{du}{u} = \ln|u| + C = \ln(t^2 + 1) + C \\ y(t) &= e^{\ln(t^2+1)+C} = A(t^2 + 1), \end{aligned}$$

where $A = e^C$. The initial condition $y(0) = 3 = A$, which implies $A = 3$. Hence, the solution is

$$y(t) = 3(t^2 + 1).$$

h. This is a separable differential equation. It can be written

$$\int e^y dy = \int e^t dt \qquad \text{or} \qquad e^y = e^t + C.$$

It follows that $y(t) = \ln(e^t + C)$. The initial condition $y(0) = 6 = \ln(1 + C)$, which implies $C = e^6 - 1$. Hence, the solution is

$$y(t) = \ln(e^t + e^6 - 1).$$

i. Rewrite this equation:

$$ye^t - 2 + (e^t - 2y)\frac{dy}{dt} = 0.$$

Since $\frac{\partial M(t,y)}{\partial y} = e^t = \frac{\partial N(t,y)}{\partial t}$, this equation is exact. Integrating we see

$$\int (ye^t - 2)dt = ye^t - 2t + h(y) \quad \text{and} \quad \int (e^t - 2y)dy = ye^t - y^2 + k(t).$$

It is clear that the potential function is

$$\phi(t, y) = ye^t - 2t - y^2 = C.$$

With the initial condition $y(0) = 6$, the solution becomes

$$\phi(t, y) = y(t)e^t - 2t - y^2(t) = 6 - 36 = -30.$$

j. The DE

$$\frac{dy}{dt} + y = y^3 e^t$$

is a Bernoulli's equation, where we make the substitution $u = y^{1-3} = y^{-2}$, so $\frac{du}{dt} = -2y^{-3}\frac{dy}{dt}$. Multiplying the above equation by $-2y^{-3}$, we obtain the linear DE in $u(t)$

$$-2y^{-3}\frac{dy}{dt} - 2y^{-2} = -2e^t \qquad \text{or} \qquad \frac{du}{dt} - 2u = -2e^t.$$

This has the integrating factor $\mu(t) = e^{-2t}$, so

$$\frac{d}{dt}\left(e^{-2t}u(t)\right) = -2e^{-t} \qquad \text{or} \qquad e^{-2t}u(t) = 2e^{-t} + C.$$

It follows that

$$\frac{1}{y^2(t)} = u(t) = 2e^t + Ce^{2t}, \qquad \text{so} \qquad 1 = 2 + C \qquad \text{or} \qquad C = -1.$$

Thus,

$$y(t) = \frac{1}{\sqrt{2e^t - e^{2t}}}.$$

2. a. The solution to the white lead problem is $P(t) = 10e^{-kt}$, where $t = 0$ represents 1970. From the data at 1975, we have $8.5 = 10e^{-5k}$ or $e^{5k} = 10/8.5 = 1.17647$. Thus, $k = 0.032504$ yr^{-1}. To find the half-life, we compute $5 = 10e^{-kt}$, so $t = \ln(2)/k = 21.33$ yr is the half-life of lead-210.

b. The differential equation can be written $P' = -k(P - r/k)$, so we make the substitution $z(t) = P(t) - r/k$. This leaves the initial value problem

$$z' = -kz, \quad z(0) = P(0) - r/k = 10 - r/k,$$

which has the solution $z(t) = (P(0) - r/k)e^{-kt} = P(t) - r/k$. Thus, the solution is

$$P(t) = \left(10 - \frac{r}{k}\right) e^{-kt} + \frac{r}{k} = 2.3086e^{-kt} + 7.6914,$$

where $k = 0.032504$. In the limit,

$$\lim_{t\to\infty} P(t) = 7.6914 \text{ disintegrations per minute of } {}^{210}\text{Pb}.$$

3. a. The differential equation describing the temperature of the tea satisfies

$$H' = -k(H - 21), \quad H(0) = 85 \text{ and } H(5) = 81.$$

Make the substitution $z(t) = H(t) - 21$, which gives the differential equation

$$z' = -kz, \quad z(0) = H(0) - 21 = 64.$$

The solution becomes $z(t) = 64e^{-kt} = H(t) - 21$ or

$$H(t) = 64e^{-kt} + 21.$$

To find k, we solve $H(5) = 81 = 64e^{-5k} + 21$ or $e^{5k} = 64/60 = 1.0667$. Thus, $k = 0.012908$ min^{-1}. The water was at boiling point when $64e^{-kt} + 21 = 100$ or $e^{-kt} = 79/64$. It follows that $t = -\ln(79/64)/k = -16.3$ min. This means that the talk went 16.3 min over its scheduled ending.

b. To obtain a temperature of at least 93°C, then we need to find the time that satisfies $H(t) = 93 = 64e^{-kt} + 21$, so $e^{-kt} = 72/64 = 1.125$. Solving for t gives $t = -\ln(72/64)/k = -9.125$ min. It follows that you must arrive at the hot water within $16.3 - 9.1 = 7.2$ min of the scheduled end of the talks.

4. a. Substituting the parameters into the differential equation gives

$$c' = \frac{1}{10^6}(22000 - 2000c) = -0.002(c - 11).$$

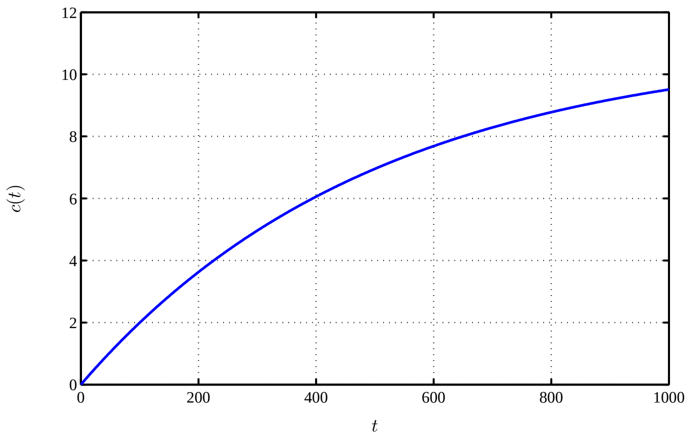
We make the substitution $z(t) = c(t) - 11$, which gives the initial value problem $z' = -0.002z$ with $z(0) = c(0) - 11 = -11$. The solution of this differential equation is $z(t) = -11e^{-0.002t} = c(t) - 11$, so

$$c(t) = 11 - 11e^{-0.002t}.$$

b. Solve the equation $c(t) = 11 - 11e^{-0.002t} = 5$, so $e^{0.002t} = 11/6$ or $t = 500\ln(11/6) = 303.1$ days. The limiting concentration

$$\lim_{t\to\infty} c(t) = 11.$$

The graph is below.



Problem 4

5. The differential equation is separable, so write

$$\int T^{-\frac{1}{2}} dT = k \int dt \qquad \text{or} \qquad 2\,T^{\frac{1}{2}}(t) = kt + C.$$

It follows that

$$T(t) = \left(\frac{kt+C}{2}\right)^2.$$

The initial condition $T(0) = 1$ implies $C = 2$, so $T(t) = \left(\frac{kt}{2} + 1\right)^2$. Since $T(4) = \left(\frac{4\,k}{2} + 1\right)^2 = 25$, $2\,k + 1 = 5$ or $k = 2$. Thus, the solution for the spread of the disease in this orchard is

$$T(t) = (t+1)^2.$$

When $t = 10$, $T(10) = 121$.

6. The differential equation with the information in the problem is given by:

$$\frac{dH}{dt} = -k(H - 25), \qquad H(0) = 35,$$

where $t = 0$ is 7 AM. We make the change of variables $z(t) = H(t) - 25$, so $z(0) = 10$. The problem now becomes

$$\frac{dz}{dt} = -kz, \qquad z(0) = 10,$$

which has the solution

$$z(t) = 10\,e^{-kt} \quad \text{or} \quad H(t) = 25 + 10\,e^{-kt}.$$

From the information at 9 AM, we see

$$H(2) = 33.5 = 25 + 10\,e^{-2k} \quad \text{or} \quad e^{2k} = \frac{10}{8.5} \quad \text{or} \quad k = \frac{\ln\left(\frac{10}{8.5}\right)}{2} = 0.081259.$$

It follows that

$$H(t) = 25 + 10\,e^{-0.081259t}.$$

The time of death is found by solving

$$H(t_d) = 39 = 25 + 10\,e^{-0.081259t_d} \quad \text{or} \quad e^{-0.081259t_d} = \frac{14}{10} \quad \text{or} \quad t_d = -\frac{\ln(1.4)}{0.081259} = -4.1407.$$

It follows that the time of death is 4 hours and 8.4 min before the body is found, which gives the time of death around 2:52 AM.

b. This differential equation is separable, so we can write:

$$\begin{aligned}
\int (H-25)^{-2/3}dH &= -k_b \int dt = -k_b t + C, \\
3(H-25)^{1/3} &= -k_b t + C, \\
H(t) &= 25 + \left(\frac{C - k_b t}{3}\right)^3.
\end{aligned}$$

The initial temperature of the body gives:

$$35 = 25 + \left(\frac{C}{3}\right)^3 \qquad \text{or} \qquad C = 3(10)^{1/3} \approx 6.4633.$$

From the temperature at $t = 2$,

$$33.5 = 25 + \left(10^{1/3} - \tfrac{2}{3}k_b\right)^3 \qquad \text{or} \qquad 8.5^{1/3} = 10^{1/3} - \tfrac{2}{3}k_b,$$

so

$$k_b = 1.5\left(10^{1/3} - 8^{1/3}\right) \approx 0.17041.$$

It follows that the time of death satisfies:

$$39 = 25 + \left(10^{1/3} - \tfrac{t_d}{3}k_b\right)^3 \qquad \text{or} \qquad 10^{1/3} - 14^{1/3} = \frac{t_d}{3}k_b.$$

Thus,

$$t_d = \frac{3}{k_b}\left(10^{1/3} - 14^{1/3}\right) \approx -4.5016 \qquad \text{or} \qquad 4 \text{ hr } 30.1 \text{ min},$$

which is approximately 2:29.9 AM. These models differ about 22 min in their predictions for the time of death.

7. a. The solution of the Malthusian growth model is $B(t) = 1000\,e^{0.01\,t}$. The population doubles when the bacteria reaches 2000, so $1000\,e^{0.01\,t} = 2000$ or $e^{0.01\,t} = 2$. Thus, $0.01\,t = \ln(2)$ or $t = 100\ln(2) \approx 69.3$ min for the population to double.

b. The model with time-varying growth is a linear and separable differential equation, so

$$\frac{dB}{dt} = 0.01(1 - e^{-t})\,B \quad \text{or} \quad \int \frac{dB}{B} = 0.01\int (1 - e^{-t})dt$$

$$\ln|B(t)| = 0.01(t + e^{-t}) + C \quad \text{or} \quad B(t) = A\, e^{0.01(t+e^{-t})},$$

where $A = e^C$. With the initial condition, $B(0) = 1000 = A\, e^{0.01}$ or $A = 1000\, e^{-0.01}$. Thus, the solution to this time-varying growth model is

$$B(t) = 1000\, e^{0.01(t+e^{-t}-1)}.$$

c. The Malthusian growth model gives $B(5) = 1051$ and $B(60) = 1822$, while the modified growth model gives $B(5) = 1041$ and $B(60) = 1804$.

8. a. The solution to the Malthusian growth model is given by $P(t) = 100\, e^{0.2\, t}$. This population doubles when $100\, e^{0.2\, t} = 200$ or $e^{0.2\, t} = 2$, so $t = 5\ln(2) \approx 3.466$ yrs.

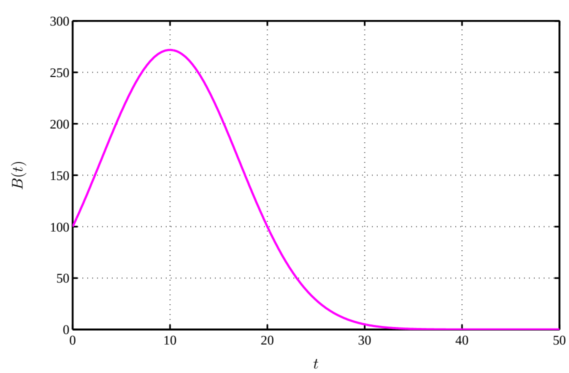
b. This model, including the modification for habitat encroachment, is a linear and separable differential equation. It can be written

$$\int \frac{dP}{P} = \int (0.2 - 0.02t)dt \qquad \text{or} \qquad \ln|P| = 0.2\, t - 0.01\, t^2 + C.$$

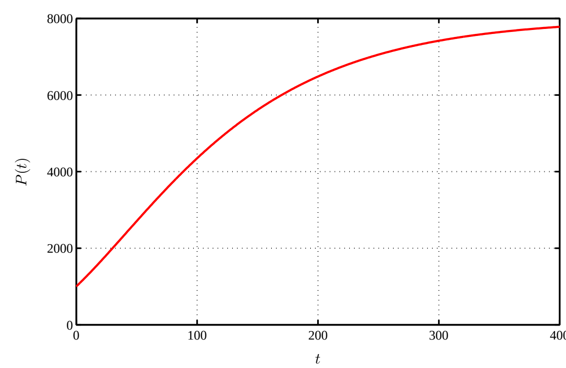
It follows that $P(t) = e^{0.2\, t - 0.01\, t^2 + C} = Ae^{0.2\, t - 0.01\, t^2}$, where $A = e^C$. The initial condition $P(0) = 100 = A$, which implies $A = 100$. Hence, the solution satisfies

$$P(t) = 100\, e^{0.2\, t - 0.01\, t^2}.$$

c. We examine the differential equation in Part b and see that $\frac{dP}{dt} = 0$ when $0.2 - 0.02t = 0$, which implies that $t = 10$. Thus, the maximum of population is $P(10) = 100\, e \approx 271.8$. If we solve $P(t) = 100\, e^{0.2\, t - 0.01\, t^2} = 100$, then this is equivalent to $e^{0.2\, t - 0.01\, t^2} = 1$ or $0.2\, t - 0.01\, t^2 = -0.01\, t(t - 20) = 0$. Thus, either $t = 20$ (or 0), so the population returns to 100 after 20 years. The graph of the population can be seen below.



Problem 8



Problem 9

9. a. This population of cells in a declining medium satisfies a separable differential equation, which can be written

$$\int P^{-2/3} dP = \int 0.3\, e^{-0.01t} dt \qquad \text{or} \qquad 3\, P^{1/3}(t) = -30\, e^{-0.01t} + 3C.$$

It follows that $P^{1/3}(t) = -10\, e^{-0.01t} + C$, so $P(t) = \left(C - 10\, e^{-0.01t}\right)^3$. The initial condition $P(0) = 1000 = (C - 10)^3$, which implies $C = 20$. The solution is given by

$$P(t) = \left(20 - 10e^{-0.01\, t}\right)^3.$$

b. This population doubles when $P(t) = (20 - 10e^{-0.01\,t})^3 = 2000$, so $20 - 10e^{-0.01\,t} = 10\sqrt[3]{2}$ or $e^{-0.01\,t} = 2 - \sqrt[3]{2}$. It follows that $t = 100\,\ln\left(\frac{1}{2-\sqrt[3]{2}}\right) \approx 30.1$ hr. For large t, $\lim_{t\to\infty} e^{-0.01\,t} = 0$, so $\lim_{t\to\infty} P(t) = 20^3 = 8000$. Thus, there is a horizontal asymptote at $P = 8000$, so the population tends towards this value. The graph of the population can be seen above.

10. a. The change in amount of phosphate, $P(t)$, is found by adding the amount entering and subtracting the amount leaving.

$$\frac{dP}{dt} = 200 \cdot 10 - 200 \cdot c(t),$$

where $c(t)$ is the concentration in the lake with $c(t) = P(t)/10{,}000$. By dividing the equation by the volume, the concentration equation is given by

$$\frac{dc}{dt} = 0.2 - 0.02c = -0.02(c - 10), \qquad c(0) = 0.$$

With the substitution $z(t) = c(t) - 10$, the equation above reduces to the problem

$$\frac{dz}{dt} = -0.02z, \qquad z(0) = -10,$$

which has the solution $z(t) = -10\,e^{-0.02\,t}$. Thus, the concentration is given by

$$c(t) = 10 - 10\,e^{-0.02\,t}.$$

b. The differential equation describing the growth of the algae is given by

$$\frac{dA}{dt} = 0.5(1 - e^{-0.02\,t})A^{2/3}.$$

By separating variables, we see

$$\begin{aligned} \int A^{-2/3} dA &= 0.5\int (1 - e^{-0.02\,t})dt \\ 3\,A^{1/3}(t) &= 0.5(t + 50\,e^{-0.02\,t}) + C \\ A(t) &= \left(\frac{0.5(t + 50\,e^{-0.02\,t}) + C}{3}\right)^3 \end{aligned}$$

From the initial condition $A(0) = 1000$, we have $1000 = \left(\frac{25+C}{3}\right)^3$. It follows that $C = 5$, so

$$A(t) = \left(\frac{t + 50\,e^{-0.02\,t} + 10}{6}\right)^3.$$

11. a. Write the differential equation $\frac{dw}{dt} = -0.2(w - 80)$, then $z(t) = w(t) - 80$. It follows that

$$\frac{dz}{dt} = -0.2z, \qquad z(0) = -80,$$

with the solution $z(t) = -80e^{-0.2t} = w(t) - 80$. Thus,

$$w(t) = 80\left(1 - e^{-0.2t}\right).$$

For a 40 kg alligator, $w(t) = 40 = 80\left(1 - e^{-0.2t}\right)$ or $40 = 80e^{-0.2t}$, so $e^{0.2t} = 2$ or $0.2t = \ln(2)$. Thus, $t = 5\ln(2) \approx 3.47$ years.

b. The pesticide accumulation is given by

$$\frac{dP}{dt} = 600\left(80\left(1 - e^{-0.2t}\right)\right), \qquad P(0) = 0.$$

The solution is given by

$$P(t) = 48,000 \int \left(1 - e^{-0.2t}\right) dt = 48,000\left(t + 5e^{-0.2t}\right) + C.$$

The initial condition gives $P(0) = 0 = 240,000 + C$, so $C = -240,000$. Hence,

$$P(t) = 48,000\left(t + 5e^{-0.2t}\right) - 240,000.$$

The amount of pesticide in the alligator at age 5 is $P(5) = 48,000\left(5 + 5e^{-1}\right) - 240,000 = 240,000e^{-1} \approx 88291$ μg.

c. The pesticide concentration for a 5 year old alligator is

$$c(5) = \frac{P(5)}{1000w(5)} = \frac{88,291}{80,000\left(1 - e^{-1}\right)} \approx 1.75 \text{ ppm}.$$

12. a. The differential equation can be written:

$$\frac{dc}{dt} = -0.004(c - 15),$$

so we make the substitution $z(t) = c(t) - 15$. Since $c(0) = 0$, it follows that $z(0) = -15$. The solution of the substituted equation is given by:

$$\begin{aligned} z(t) &= -15e^{-0.004t} = c(t) - 15 \\ c(t) &= 15 - 15e^{-0.004t}. \end{aligned}$$

The limiting concentration satisfies:

$$\lim_{t \to \infty} c(t) = 15 \text{ mg/m}^3.$$

b. We begin by separating variables, which gives:

$$\begin{aligned} \int \frac{dc}{c - 15} &= -0.001 \int (4 - \cos(0.0172t))\, dt \\ \ln(c(t) - 15) &= -0.001\left(4t - \frac{\sin(0.0172t)}{0.0172}\right) + C \\ c(t) &= 15 + Ae^{-0.001\left(4t - \frac{\sin(0.0172t)}{0.0172}\right)} \end{aligned}$$

It is easy to see that the initial condition $c(0) = 0$ implies that $A = -15$. Thus, the solution to this problem is given by:

$$c(t) = 15 - 15\, e^{-0.001(4t - 58.14\sin(0.0172t))}$$

13. a. We separate variables, so

$$\int M^{-3/4} dM = -k \int dt \quad \text{or} \quad 4M^{1/4} = -kt + 4C$$

$$M(t) = \left(C - \frac{k}{4}t\right)^4$$

From the initial condition, $M(0) = 16 = C^4$, it follows that $C = 2$. From the information that $M(10) = 1 = (2 - 10k/4)^4$, we have $k = 0.4$, so

$$M(t) = (2 - 0.1t)^4.$$

The fruit vanishes in 20 days.

b. We separate variables again to find:

$$\int M^{-3/4} dM = -0.8 \int e^{-0.02t} dt \quad \text{or} \quad 4M^{1/4} = \frac{0.8}{0.02} e^{-0.02t} + 4C$$

$$M(t) = \left(10e^{-0.02t} + C\right)^4.$$

From the initial condition, $M(0) = 16 = (10 + C)^4$, it follows that $C = -8$, so

$$M(t) = \left(10e^{-0.02t} - 8\right)^4.$$

Solving $10e^{-0.02t} = 8$, which is when the fruit vanishes, we find $t = 50\ln(5/4)$. Thus, the fruit vanishes in 11.157 days.

14. a. The general solution to the Malthusian growth problem with the initial condition $P(0) = 60$ is

$$P(t) = 60\, e^{rt}.$$

We are given that 2 weeks later $P(2) = 80 = 60\, e^{2r}$, so it follows that $r = \frac{1}{2}\ln\left(\frac{4}{3}\right) = 0.14384$. This gives the solution:

$$P(t) = 60\, e^{0.14384\, t}.$$

It is easy to see that the population doubles when $120 = 60\, e^{0.14384\, t}$, so $0.14384\, t_d = \ln(2)$ or the doubling time is

$$t_d = \frac{\ln(2)}{r} = 4.819 \text{ weeks}.$$

b. We begin by separating variables, so the general solution satisfies:

$$\int \frac{dP}{P} = \int (a - b\, t)\, dt \quad \text{or} \quad \ln(P(t)) = a\, t - \frac{bt^2}{2} + C \quad \text{or} \quad P(t) = e^C e^{a\, t - \frac{bt^2}{2}}.$$

Since the initial value is $P(0) = 60$, it follows that $e^C = 60$. Thus,

$$P(t) = 60\, e^{a\, t - \frac{bt^2}{2}}.$$

We now use the data at $t = 2$ and 4 weeks. It follows from the solution above that

$$\begin{aligned} 80 &= 60\, e^{2\,a-2\,b} \\ 90 &= 60\, e^{4\,a-8\,b}. \end{aligned}$$

We rearrange the terms and take logarithms of both sides to get

$$\begin{aligned} 2\,a - 2\,b &= \ln\left(\frac{4}{3}\right) \\ 4\,a - 8\,b &= \ln\left(\frac{3}{2}\right). \end{aligned}$$

We solve these equations simultaneously to obtain

$$2\,b = \ln\left(\frac{4}{3}\right) - \frac{1}{2}\ln\left(\frac{3}{2}\right),$$

so $b = 0.042475$. But $a = b + \frac{1}{2}\ln(4/3)$ or $a = 0.1863$. It follows that the solution is

$$P(t) = 60\, e^{0.1863\,t - 0.021237\,t^2}.$$

The population reaches a maximum when the derivative is zero, which occurs when $t_{max} = \frac{a}{b} = 4.3865$, so the maximum population is $P(t_{max}) = 90.286$.

15. The differential equation, $\frac{dR}{dt} = -0.05R + 0.2e^{-0.01t}$ with $R(0) = 10$ is linear and can be written

$$\frac{dR}{dt} + 0.05R = 0.2e^{-0.01t} \qquad \text{with} \qquad \mu(t) = e^{0.05t}.$$

It follows that it can be written

$$\frac{d}{dt}\left(e^{0.05t}R(t)\right) = 0.2e^{0.04t} \qquad \text{or} \qquad e^{0.05t}R(t) = 0.2\int e^{0.04t}dt = 5e^{0.04t} + C.$$

Thus,

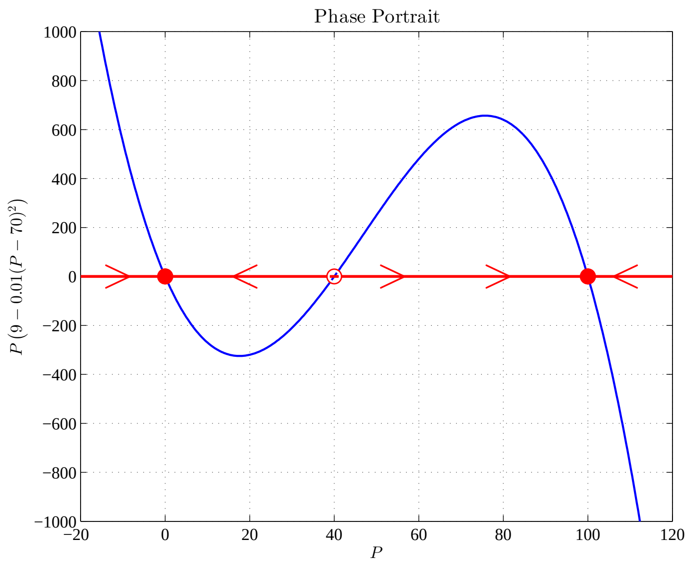
$$R(t) = 5e^{-0.01t} + Ce^{-0.05t}, \qquad \text{or} \qquad R(0) = 10 = 5 + C.$$

The solution is $R(t) = 5e^{-0.05t} + 5e^{-0.01t}$.

16. (Allee effect) Consider the DE given by the model:

$$\frac{dP}{dt} = P\left(9 - 0.01(P - 70)^2\right) = A(P).$$

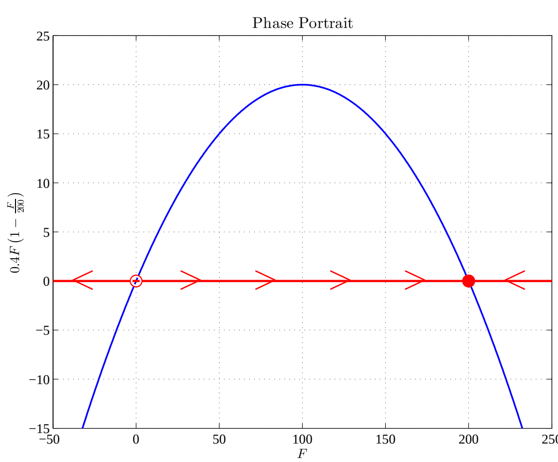
The equilibria of this population model satisfy $P\left(9 - 0.01(P - 70)^2\right) = 0$. Thus, $P_e = 0$, 40, and 100. From the phase portrait below, it is easy to see that the equilibria $P_e = 0$ and 100 are stable, while $P_e = 40$ is unstable. The carrying capacity for this population is $P_e = 100$, and the critical threshold number of animals required to avoid extinction is $P_e = 40$.



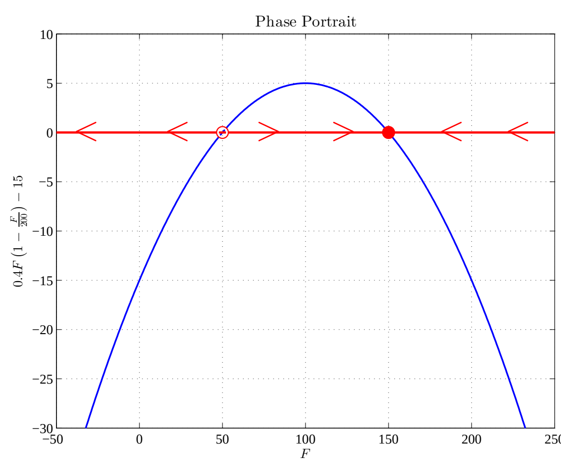
17. a. The solution follows the logistic growth solution seen in 1. d. The solution is

$$F(t) = \frac{10,000}{50 + 150e^{-0.4t}}.$$

b. This is a standard logistic growth model, so the equilibria are $F_e = 0$ and 200 (thousand). Below is a sketch of the function with the phase portrait. The equilibrium $F_e = 0$ is unstable, while the carrying capacity, $F_e = 200$ (thousand), is a stable equilibrium.



Problem 18. b



Problem 18. c

c. With harvesting, the right hand side of the differential equation is written

$$0.4\,F\left(1 - \frac{F}{200}\right) - 15 = -0.002F^2 + 0.4F - 15 = -0.002(F-50)(F-150).$$

It follows that the equilibria are $F_e = 50$ and 150 (thousand). Above is a sketch of the function with the phase portrait. The equilibrium $F_e = 50$ (thousand) is the critical number of fish needed to avoid extinction and this equilibrium is unstable. The carrying capacity, $F_e = 150$ (thousand), is a stable equilibrium.