**Homework 6 – Singular Perturbations Due Wed. 12/18**

For the inner solutions (boundary layer) of boundary value problems (BVP) we made the change of variables $\xi = x/\delta(\varepsilon)$ when the left boundary condition (BC) occurred at $x = 0$. Be sure to briefly explain your choice of $\delta(\varepsilon)$ for each BVP below. If the inner solution (boundary layer) occurs at the right BC, $x = 1$, for a BVP, then you may want to consider the change of variables $\xi = (1 - x)/\delta(\varepsilon)$.

1. a. Use a dominant balancing method to determine the leading order behavior of the roots of the following algebraic equation:

$$\varepsilon x^3 + x - 2 = 0.$$

Explain why a singular perturbation method is necessary. Find a first-order correction for the leading behavior. (Hint: You may want to assume an expansion of the form $y = x_0 + x_1\sqrt{\varepsilon} + x_2\varepsilon + \dots$) Compare the answers from this approximation for $\varepsilon = 0.01$ and $0.0001$. (You may want Maple to get the exact values of the roots.)

b. Find second and third order approximations, *i.e.*, find $x_2$ and $x_3$. Again let $\varepsilon = 0.01$ and $0.0001$ and determine how these two approximations compare to the exact roots.

2. a. Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\varepsilon y'' + 2y' + y = 0, \qquad y(0) = 0, \quad y(1) = 1, \quad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived. Find the exact solution to this BVP problem.

b. Let $\varepsilon = 0.1$ and $0.05$ and create graphs (one for each $\varepsilon$) showing the inner, outer, and uniform solutions along with the exact solution. Briefly discuss what you observe with these graphs.

3. a. Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\varepsilon y'' + y' + y^2 = 0, \qquad y(0) = \frac{1}{4}, \quad y(1) = \frac{1}{2}, \quad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived.

b. Let $\varepsilon = 0.1$ and $0.05$ and create graphs (one for each $\varepsilon$) showing the inner, outer, and uniform solutions. Briefly discuss what you observe with these graphs. Extra-credit if you can implement a BVP solver such as one of the ones in MatLab to give a numerical exact solution and add this to your graph. (Hint: The uniform solution provides a great guess for the BVP solvers.)

4. a. Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\varepsilon y'' + (1 + x)y' = 1, \qquad y(0) = 0, \quad y(1) = 1 + \ln(2), \quad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived.

b. Let $\varepsilon = 0.1$ and $0.05$ and create graphs (one for each $\varepsilon$) showing the inner, outer, and uniform solutions. Briefly discuss what you observe with these graphs. Extra-credit if you can implement a BVP solver such as one of the ones in MatLab to give a numerical exact solution and add this to your graph. (Hint: The uniform solution provides a great guess for the BVP solvers.)

5. a. Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\varepsilon u'' - (2x+1)u' + 2u = 0, \qquad u(0) = 1, \quad u(1) = 0, \quad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived.

b. Let $\varepsilon = 0.1$ and $0.05$ and create graphs (one for each $\varepsilon$) showing the inner, outer, and uniform solutions. Briefly discuss what you observe with these graphs. Extra-credit if you can implement a BVP solver such as one of the ones in MatLab to give a numerical exact solution and add this to your graph. (Hint: The uniform solution provides a great guess for the BVP solvers.)

6. a. Use singular perturbation methods to obtain a uniformly valid approximation to the solution of the initial value problem (IVP):

$$\varepsilon y'' + (t+1)^2 y' = 1, \qquad y(0) = 1, \quad \varepsilon y'(0) = 1, \quad 0 < \varepsilon \ll 1.$$

State clearly both the inner and outer solutions that you derived.

b. Let $\varepsilon = 0.1$ and $0.02$ and create graphs (one for each $\varepsilon$) for $t \in [0, 5]$ showing the inner, outer, and uniform solutions. Use a numerical ODE solver to find good approximations to the actual solution and add these to your graphs. Briefly discuss what you observe with these graphs.

7. a. Consider the initial value problem (IVP) given by the system:

$$\begin{aligned} x' &= ky - x + \varepsilon xy, \qquad & x(0) &= 1, \\ \varepsilon y' &= x - \varepsilon xy - y, \qquad & y(0) &= 0, \quad 0 < \varepsilon \ll 1. \end{aligned}$$

Use singular perturbation methods to approximate the solution. State clearly both the inner and outer solutions that you derived for both $x(t)$ and $y(t)$. Discuss why the convergence to the actual solution differs between the cases $k < 1$ and $k > 1$. Does either case provide a uniformly valid approximation for $t > 0$? Explain.

b. Let $k = 0.7$. Consider $\varepsilon = 0.1$ and $0.05$ and create four graphs, showing separately $x(t)$ and $y(t)$ (with one for each $\varepsilon$) for $t \in [0, 5]$ showing the inner, outer, and uniform solutions. Use a numerical ODE solver to find good approximations to the actual solution and add these to your graphs. Briefly discuss what you observe with these graphs.

c. Repeat the steps in Part b with $k = 1.3$. Include a comparison between the solutions in Parts b and c.