Homework 4 – Power Series Due Mon. 11/4

Work all problems in WeBWorK.

1. For the initial-value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(0) = y_0, \quad y'(0) = v_0,$$

find a transformation $y = vY$ such that Y solves

$$Y'' + Q(x)Y = 0.$$

What are $v(x)$ and $Q(x)$? What do the initial conditions become for the $Y(x)$ version of the problem?

2. For the following problems, show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions (showing at least **4** non-zero terms in each series solution).

a. $2x^2y'' + xy' + x^2y = 0$.

b. $x^2y'' + 3xy' + (1 + x)y = 0$.

c. $x^2y'' + 4xy' + (2 + x)y = 0$.

3. Bessel's equation of order $\frac{1}{2}$ is given by:

$$x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right) y = 0.$$

This equation is important in solving partial differential equations with spherical geometry.

a. Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Determine expressions for the coefficients for these two solutions.

b. Consider the change of variables, $y(x) = x^{-\frac{1}{2}}v(x)$. Show that this change of variables reduces the Bessel's equation above to a much simpler ODE in $v(x)$. Solve this problem in v and determine the relatively simple closed form solutions, y, for Bessel's equation of order $\frac{1}{2}$, $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$.

4. Consider the following ODE:

$$x^2y'' + 6xy' + \left(6 - x^2\right) y = 0. \tag{1}$$

a. Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Determine expressions for the coefficients for these two solutions.

b. Consider the change of variables, $y(x) = x^{\alpha} v(x)$. With this change of variables, find α that eliminates any term with v', reducing (1) to a much simpler ODE in $v(x)$. Solve this problem in v and determine the relatively simple closed form solutions, y, for (1). Connect this solution to your series solution.

5. Consider the following ODE:

$$xy'' + (1 - 2x)y' + (x - 1)y = 0. \tag{2}$$

a. Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions. Determine expressions for the coefficients for these two solutions.

b. **Reduction of Order** (Jean D'Alembert (1717-1783)): If $y_1(x)$ is known for the linear ODE:

$$y'' + p(x)y' + q(x)y = 0.$$

Then one attempts a solution of the form $y(x) = v(x)y_1(x)$. Provided $y_1(x) \neq 0$, show that

$$\frac{dv}{dx} = \frac{1}{[y_1(x)]^2} e^{-\int^x p(s)ds}.$$

Solve for $v(x)$ to obtain the 2^{nd} linearly independent solution, $y_2(x)$. (This will just be an integral expression, not easily simplified.)

c. In Part a, $y_1(x)$ should be easily recognizable as a basic function. Use the **Reduction of Order** method to find $y_2(x)$ for (2) and compare this solution to your series solution in Part a.

6. Consider the following ODE:

$$xy'' - (2 + x^2)y' + xy = 0. \tag{3}$$

a. Show that $x = 0$ is a regular singular point, find the indicial equation and recurrence relations, and determine the two linearly independent solutions.

b. Verify that one solution to (3) is

$$y_1(x) = e^{\frac{x^2}{2}}.$$

Does this match one of the solutions that you found in Part a? Use the **Reduction of Order** method from Problem 5 to find a second linearly independent solution.