Fall 2019 Math 537

Homework 3 – Fundamental Solutions Due Fri. 10/4

Work all problems in WeBWorK and attach any written parts to this assignment.

1. a. Use the matrix definition of e^{At} to find a fundamental matrix solution of $\dot{\mathbf{y}} = A\mathbf{y}$, where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

b. Use the matrix definition of e^{At} to find a fundamental matrix solution of $\dot{\mathbf{y}} = A\mathbf{y}$, where A is the $n \times n$ matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

2. Use the matrix definition of e^{A} to show that if P is a nonsingular $n \times n$ matrix,

$$P^{-1}e^{A}P = e^{P^{-1}AP}.$$

3. Show that if $AB = BA$ (commute), then

$$e^{A+B} = e^{A}e^{B} = e^{B}e^{A}.$$

Since A and B commute, you can make use of the binomial theorem which says

$$(A+B)^m = \sum_{l=0}^{m} \binom{m}{l} A^l B^{m-l}, \quad \binom{m}{l} = \frac{m!}{l!(m-l)!}$$

You will also need to use a change of order for the double sum that will come up. This means using the identity

$$\sum_{j=0}^{\infty}\sum_{l=0}^{j} a_{jl} = \sum_{l=0}^{\infty}\sum_{j=l}^{\infty} a_{jl}.$$

4. Find an example of two matrices such that

$$e^{A+B} \neq e^{A}e^{B}.$$

Hint: You can do this with 2×2 matrices.

5. Consider

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = A + B, \quad \text{and} \quad A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$

a. Show that A and B commute, so $e^{A+B} = e^A e^B$.

b. Use the definition of e^{Bt} to prove that

$$e^{Bt} = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$

c. From these results, give the real Jordan canonical form, e^{Jt}.

6. (Proof of Abel's formula for 2×2 case)
a. Let

$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$$

with $b_{ij}(t)$ differentiable. Compute $\frac{d}{dt}(\det B(t))$ by first expanding $\det B(t)$ and then differentiating. Next show that

$$\frac{d}{dt}(\det B(t)) = \det \begin{pmatrix} b'_{11}(t) & b'_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} + \det \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b'_{21}(t) & b'_{22}(t) \end{pmatrix}.$$

b. Let

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix}$$

be a fundamental solution for $x' = A(t)x$, where

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}.$$

Show that

$$\begin{aligned} \frac{d}{dt}(\det \mathbf{\Phi}(t)) &= \det \begin{pmatrix} \sum_{k=1}^2 a_{1k}x_{k1}(t) & \sum_{k=1}^2 a_{1k}x_{k2}(t) \\ x_{21}(t) & x_{22}(t) \end{pmatrix} + \det \begin{pmatrix} x_{11}(t) & x_{12}(t) \\ \sum_{k=1}^2 a_{2k}x_{k1}(t) & \sum_{k=1}^2 a_{2k}x_{k2}(t) \end{pmatrix}. \\ &= \sum_{i=1}^{2} a_{ii} \det \mathbf{\Phi}(t). \end{aligned}$$

c. Let $z(t) = \det \mathbf{\Phi}(t)$ where $\mathbf{\Phi}(t)$ is the fundamental solution to $x' = A(t)x$ as above. Use Part b to conclude that

$$z(t) = z(0)e^{\int_0^t (\mathrm{tr} A(s))ds},$$

thus establishing this special case of Abel's formula.

7. Consider the linear system of ODEs given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

For each of the following matrices A, find a transition matrix P that transforms A into the **real Jordan canonical form**, J. Write both P and J. Furthermore, give a **fundamental solution**, $\mathbf{\Psi}(t) = e^{Jt}$. Use this solution to write the **unique solution**, $\mathbf{x}(t)$, to the initial value problem above.

$$A = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix} \quad \mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -6 & -7 & -4 \end{pmatrix} \quad \mathbf{x}_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

8. a. For real paramaters a, b, and c, describe the regions in (a, b, c) space where the matrix

$$A = \begin{pmatrix} a & 0 & 0 & a \\ 0 & a & b & 0 \\ 0 & c & a & 0 \\ a & 0 & 0 & a \end{pmatrix}$$

has real, complex, and repeated eigenvalues. **Hint:** Breaking A up into four 2×2 sub-matrices, or blocks, makes your life significantly easier. So use

$$A = \begin{pmatrix} aI_2 & A_{12} \\ A_{21} & aI_2 \end{pmatrix}, \; I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_{12} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \; A_{21} = \begin{pmatrix} 0 & c \\ a & 0 \end{pmatrix}.$$

b. Let $a = b = c = 2$. Transform A into the **real Jordan canonical form**, J, writing both P and J. Then give a **fundamental solution**, $\mathbf{\Psi}(t) = e^{Jt}$, solving the system of ODEs

$$\dot{\mathbf{\Psi}} = J\mathbf{\Psi}.$$

9. Find $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$ for

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Hint: The $\|A\|_2$ can be found taking $\mathbf{x} = [\cos(\theta), \sin(\theta)]^T$ and finding the max $\|A\mathbf{x}\|_2$ over θ (a Calculus problem).

10. If A is an invertible matrix, show that

$$\|A\|\|A^{-1}\| \geq 1.$$