1. a. The first model has constants $k = 0.084582$ and $a = 2.2542$ with the $SSE = 5.8156$.

b. The second model has constants $k = 0.094524$ and $a = 2.2145$ with the $SSE = 4.9664$.

c. The third model uses $a = 2$, then $k = 0.17536$ with the $SSE = 27.2195$.

d. Part a uses the stable, easy linear least squares formula to find a and k, but the logarithms weight the early data more. Part b has the best non-biased fit to the data, but the nonlinear fitting routine could have instabilities and is certainly more complex. Part c uses the dimensional analysis best fit with the $a = 2$, so is better connected to the physical model, and this fit uses only a 1D parameter search, which is very stable. However, the data fit is significantly poorer than the other two models. The value of $a = 2$ occurs because the strength of a tree occurs from its cross-sectional area, which scales as diameter squared.

2. a. Because the data is spaced, this problem requires solving by simulating the model from $n = 0$ to 15 and fitting the best parameters P_0, r, and M to the logistic growth model:

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right), \qquad P(0) = P_0.$$

The best fitting parameters are $P_0 = 2.3583$, $r = 0.61524$, and $M = 231.35$ with the $SSE = 123.29$. In this model, the parameter r reflects the Malthusian growth, while M is the carrying capacity of the population.

b. The logistic growth model always has equilibria, $P_{1e} = 0$ (unstable) and $P_{2e} = M = 231.35$ (stable). The derivative satisfies:

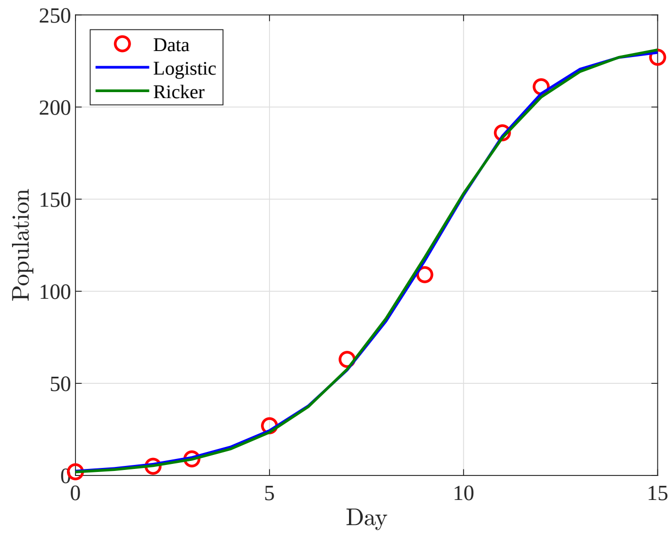
$$F'(P) = 1.61524 - 0.0053187P.$$

Thus, $F'(0) = 1.61524 > 1$, showing solutions near $P_{1e} = 0$ grow monotonically away from this equilibrium (unstable). $F'(M) = 0.38476 < 1$, showing solutions near $P_{2e} = 0$ monotonically approach this equilibrium (stable). From a modeling perspective, this analysis shows that any reasonable positive population ($0 < P_0 < 607$) will monotonically approach the carrying capacity of 231.35.

c. For Ricker's model with

$$P_{n+1} = aP_ne^{-bP_n}, \qquad P(0) = P_0,$$

the best fitting parameters are $P_0 = 1.9005$, $a = 1.67712$, and $b = 0.0022002$ with the $SSE = 185.38$. In this model, the parameter $a = 1 + r$, where r reflects the Malthusian growth. The parameter b limits the growth rate and reflects the carrying capacity through $P_{2e} = \ln(a)/b = 235.01$. We see that this best fitting Malthusian growth rate is about 9% higher than the one for the logistic growth, while the parameter b produces a very similar carrying capacity. Below is a graph showing the best fitting logistic and Ricker's growth models with the data. (This graph was not requested, but shown for completeness.)

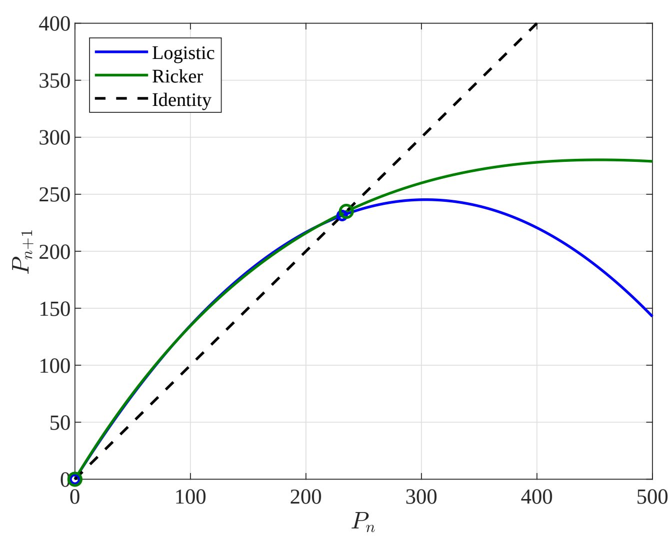


d. Ricker's growth model always has equilibria, $P_{1e} = 0$ (unstable) and $P_{2e} = \frac{\ln(a)}{b} = 235.01$ (stable). The derivative satisfies:

$$R'(P) = (1.67712 - 0.0036900\,P)e^{-0.0022002P}.$$

Thus, $R'(0) = 1.67712 > 1$, showing solutions near $P_{1e} = 0$ grow monotonically away from this equilibrium (unstable). $R'(235.01) = 0.48292 < 1$, showing solutions near $P_{2e} = 0$ monotonically approach this equilibrium (stable). From a modeling perspective, this analysis shows that any positive population will monotonically approach the carrying capacity of 235.01.

e. Below is a graph of the updating functions for both the logistic and Ricker's model, including the identity map and the equilibria where the updating functions intersect the identity map.



The maximum growth rate for the logistic growth model occurs at $(303.69, 245.26)$. The maximum growth rate for the Ricker's growth model occurs at $(454.50, 280.42)$. The logistic updating function becomes negative (unrealistic) after $P_n > 607.4$, while Ricker's updating function decreases after the maximum but goes to a horizontal asymptote of $P_{n+1} = 0$. In the domain of the data and before achieving the carrying capacity equilibrium, the two updating functions remain extremely close, suggesting very similar behavior, which is observed in the time series graph above.

f. We see that the logistic growth model has a smaller sum of square errors, so it fits the data best. However, visually there is little difference between these models. The updating functions vary significantly after they pass through the larger equilibrium. Significantly, for large populations Ricker's model remains positive. The strength of the logistic model is its simplicity and good fit to the data. Its primary weakness is that the quadratic goes negative for larger populations. Ricker's model also fits the data well with only the two dynamic parameters, so is equally simple, but it behaves better for large populations. It does not have a significant weakness compared to the logistic, but it does fail to fit the data as well and is a more complicated expression than a simple quadratic function.

3. a. The model given by

$$C_{n+1} = \alpha C_n + \mu,$$

has the best fitting initial value $C_0 = 0.0028169$, while the best fitting parameters are $\alpha = 0.26915$ and $\mu = 1.8485$. The sum of square errors is 0.0024580. The model satisfies $c(2) = 2.34624$ with a percent error of 1.569%. It satisfies $c(5) = 2.52569$ with a percent error of -0.9534%. The equilibrium for this model is $C_e = 2.52926$. The parameter μ relates to the amount drug injected daily (divided by the volume of the blood compartment of the body to give concentration) and α is the fraction remaining from the previous day after the drug has been either metabolized or excreted from the body. ($1 - \alpha$ is the rate of metabolism or excretion.) The equilibrium, C_e, is the steady-state concentration of the drug in the body, which is needed to be at a level that is therapeutic.

b. The solution to the ODE model is given by:

$$c(t) = c_\infty + (c_0 - c_\infty)\, e^{-bt},$$

has the best fitting parameters $c_\infty = 2.52925$, $c_0 = 0.0028148$, and $b = 1.3125$. The sum of square errors is 0.0024580. (Note that this model converges better than the discrete model.) This model satisfies $c(2) = 2.34624$ with a percent error of 1.569%. It satisfies $c(5) = 2.52568$ with a percent error of -0.9536%. This model is essentially the same as the model in Part a using continuous variables. In this model, $c_\infty = 2.52925$ is the equilibrium or homeostasis level of the drug, while b is the rate that the drug is metabolized and/or excreted. Once again, the equilibrium, $c_\infty = 2.52925$ is the steady-state concentration of the drug in the body.

4. a. With the logistic growth models of the form:

$$\begin{aligned} \frac{dX}{dt} &= r_x X\left(1 - \frac{X}{M_x}\right), \qquad X(0) = X_0, \\ \frac{dY}{dt} &= r_y Y\left(1 - \frac{Y}{M_y}\right), \qquad Y(0) = Y_0, \end{aligned}$$

the solution to the first model is

$$X(t) = \frac{X_0 M_x}{X_0 + (M_x - X_0)e^{-r_x t}},$$

which is fit to the given data. The best fitting initial conditions and parameters are for Species X: $X_0 = 12.1869$, $M_x = 173.2115$, $r_x = 0.069787$, and for Species Y: $Y_0 = 5.14821$, $M_y = 110.957$, $r_y = 0.035263$. The least sums of square errors between the data and the models are $SSE_X = 18.293$ and $SSE_Y = 3.5750$.

b. The competition model has the form:

$$\begin{aligned} \frac{dX}{dt} &= r_x X\left(1 - \frac{X}{M_x}\right) - \frac{XY}{a_3}, \\ \frac{dY}{dt} &= r_y Y\left(1 - \frac{Y}{M_y}\right) - \frac{XY}{b_3}, \end{aligned}$$

where the constants r_x, r_y, M_x, and M_y are the same as above. The best initial conditions and parameters satisfy: $X_0 = 8.98265$, $Y_0 = 6.29768$, $a_3 = 6451.93$, and $b_3 = 8895.61$. The least sums of square errors between the data and the models are $SSE_X = 11.577$

c. There are **4** equilibria: $(X_e, Y_e) = (0, 0)$, $(173.2115, 0)$, $(0, 110.957)$, and $(151.0860, 57.5148)$. The Jacobian matrix satisfies:

$$J(X,Y) = \begin{pmatrix} 0.069787 - 0.000805801X - 0.000154992Y & -0.000154992X \\ -0.000112415Y & 0.035263 - 0.000635616Y - 0.000112415X \end{pmatrix}.$$

At the equilibrium $(X_e, Y_e) = (0, 0)$, the Jacobian matrix is:

$$J(0,0) = \begin{pmatrix} 0.069787 & 0 \\ 0 & 0.035263 \end{pmatrix},$$

which has eigenvalue $\lambda_1 = 0.069787$ with eigenvector $\xi_1 = [1, 0]^T$ and $\lambda_2 = 0.035263$ with eigenvector $\xi_2 = [0, 1]^T$. This is an unstable node.

At the equilibrium $(X_e, Y_e) = (173.2115, 0)$, the Jacobian matrix is:

$$J(173.2115, 0) = \begin{pmatrix} -0.069787 & -0.026846 \\ 0 & 0.015791 \end{pmatrix},$$

which has eigenvalue $\lambda_1 = -0.069787$ with eigenvector $\xi_1 = [1, 0]^T$ and $\lambda_2 = 0.015791$ with eigenvector $\xi_2 = [-0.31371, 1])]^T$. This is a saddle point.

At the equilibrium $(X_e, Y_e) = (0, 110.957)$, the Jacobian matrix is:

$$J(0, 110.957) = \begin{pmatrix} 0.052590 & 0 \\ -0.012473 & -0.035263 \end{pmatrix},$$

which has eigenvalue $\lambda_1 = 0.052590$ with eigenvector $\xi_1 = [1, -0.14198]^T$ and $\lambda_2 = -0.035263$ with eigenvector $\xi_2 = [0, 1]^T$. This is a saddle point.

At the equilibrium $(X_e, Y_e) = (151.0860, 57.5148)$, the Jacobian matrix is:

$$J(151.0860, 57.5148) = \begin{pmatrix} -0.060873 & -0.023417 \\ -0.0064655 & -0.018279 \end{pmatrix},$$

which has eigenvalue $\lambda_1 = -0.014980$ with eigenvector $\xi_1 = [0.45451, -0.89074]^T$ and $\lambda_2 = -0.064172$ with eigenvector $\xi_2 = [0.99022, 0.13951]^T$. This is a stable node.

d. The graph below is produced by pplane8, showing the phase portrait with nullclines and equilibria and including the trajectory starting near the initial experimental point. It is clear that any positive initial condition will result in the populations eventually approaching the coexistence equilibrium at $(X_e, Y_e) = (151.0860, 57.5148)$.