1. The data below came from the Allegheny National Forest in Pennsylvania.

d	V	d	V	d	V
8.4	10.3	11.5	21.2	17.5	55.3
11	18.3	13	27.3	20.7	76.7

The diameter, d, is in inches, and the volume, V, is in board feet. We want to examine **3** models of the form:

$$V = kd^a, \tag{1}$$

for some constants k and a.

a. In the first model, find the constants k and a that best fit the allometric model, using a linear fit to the logarithms of the data. Also, determine the sum of square errors.

b. In the second model, find the constants k and a that give the nonlinear least squares best fit to the data. Write this sum of square errors.

c. In the third model, let a be the integer value closest to the models above, then find the nonlinear least squares fit to the data by only changing k and give this sum of square errors.

d. List a strength and weakness for each of the models above. Explain why a in the third model has the value given.

2. When a monoculture of an organism is grown in a limited (but renewed) medium, then the population of that organism often follows the logistic growth model. Below is a table for the population of a fresh water organism.

Day	0	2	3	5	7	9	11	12	15
Population(/cc)	2	5	9	27	63	109	186	211	227

a. Consider the discrete logistic growth model

$$P_{n+1} = P_n + rP_n\left(1 - \frac{P_n}{M}\right), \qquad P(0) = P_0.$$

Use the data above to find the parameters P_0, r, and M that best fit this model. Write the least sum of square errors. Briefly discuss the meaning of each of the parameters and what each term in the model represents with respect to population dynamics.

b. Find all equilibria for the logistic growth model with the parameters found above. Determine the stability of all the equilibria (include the values of the derivative). From a modeling perspective, describe what this analysis means about the equilibria.

c. Another important model used in ecology is the Ricker's model given by

$$P_{n+1} = aP_ne^{-bP_n}, \qquad P(0) = P_0.$$

Use the data above to find the parameters P_0, a, and b that best fit this model. Write the least sum of square errors. Briefly discuss what each of these parameters represent and compare them to the parameters in the logistic growth model.

d. Find all equilibria for the Ricker's growth model with the parameters found above. Determine the stability of all the equilibria (include the values of the derivative). From a modeling perspective, describe what this analysis means about the equilibria.

e. Sketch a graph the updating functions for both of these models. Include the identity map in your graph, showing clearly points of intersection with the updating functions. Find the maximum growth rate for each of the models, including the population at this maximum growth rate. Qualitatively, what are the main differences between each of these updating functions? Are there significant differences in the domain where the data are collected?

f. Which model fits the data best or are the models very similar? Give a reason why ecologists might prefer the Ricker's model to the logistic model for population studies. List at least one strength of each model. List at least one weakness with each model.

3. Drugs are used for a wide variety of treatments in the U. S. Because of toxicity of a drug, the drug is given in doses that build up in the body to a therapeutic dose. The dose is designed to be sufficiently high to be effective, but not so high as to be toxic. Suppose that a specific drug is monitored in a patient for its concentration in the blood (in ng/ml of blood serum). The table below shows the time course of the drug in this patient for the first five days.

Day	Concentration	Day	Concentration
0	0	3	2.47
1	1.87	4	2.52
2	2.31	5	2.55

a. A drug entering the body is usually in a dynamic state of flux, which complicates maintenance of a therapeutic dose. A discrete dynamical model can be used to track the amount of drug in the body and has the following form:

$$C_{n+1} = \alpha C_n + \mu,$$

where C_0 is the initial concentration and α and μ are parameters for the model. Describe how the parameters α and μ relate to administration and metabolism or excretion of the drug in the patient. Simulate this model from $t = 0$ to 5 and find the best fitting values for C_0, α, and μ. Give the sum of square errors. Find the percent error at $t = 2$ and 5. Find the equilibrium for this model and describe how this relates to the drug in this particular patient.

b. An alternate model can be described by the differential equation given by:

$$\frac{dc}{dt} = b\left(c_\infty - c\right), \qquad c(0) = c_0,$$

where the parameters c_∞, b, and c_0 are fit to the data using a nonlinear least squares best fit. Solve this initial value problem. Describe how this model relates to administration and metabolism or excretion of the drug in this patient. Determine the best fitting parameters and give the sum of square errors. Find the percent error at $t = 2$ and 5. Find the equilibrium for this model and describe how this relates to the drug in this particular patient.

4. a. Two species of yeast (X and Y) are placed in a chemostat with a constant supply of nutrients. Data are first collected on the monocultures of each species of yeast in this environment and are summarized in the table below:

t (hr)	Yeast X	Yeast Y
0	12.1	5.2
5	16.3	6.5
10	23.4	6.9
20	40.5	10.3
30	66.8	13.8
40	94.3	18.0
50	123.7	24.7
60	145.2	31.8
70	155.4	39.4
80	167.8	51.2
90	169.1	59.6
100	169.7	68.9

These growth conditions satisfy the assumptions for logistic growth models of the form:

$$\begin{aligned}
\frac{dX}{dt} &= r_x X\left(1 - \frac{X}{M_x}\right), \qquad X(0) = X_0, \\
\frac{dY}{dt} &= r_y Y\left(1 - \frac{Y}{M_y}\right), \qquad Y(0) = Y_0.
\end{aligned}$$

Use the data above to find the best initial conditions and parameters, X_0, r_x, M_x, Y_0, r_y, and M_y to fit these models. Also, give the sums of square errors between the data and the models.

b. A new experiment on competition of the two species of yeast is performed where the two species are introduced into a single chemostat and allowed to grow together. Below is a table of the results for the competition experiment.

t (hr)	X	Y
0	9.2	6.4
5	12.7	7.5
10	17.6	8.8
20	30.2	11.3
30	51.7	14.8
40	77.6	19.2
50	102.4	22.6
60	126.5	27.3
70	143.5	28.4
80	149.7	32.2
90	154.6	35.1
100	157.2	36.9

A competition model is created using the information from the single species experiments. The model has the form:

$$\begin{aligned}\frac{dX}{dt} &= r_x X\left(1-\frac{X}{M_x}\right)-\frac{XY}{a_3},\\ \frac{dY}{dt} &= r_y Y\left(1-\frac{Y}{M_y}\right)-\frac{XY}{b_3}.\end{aligned}$$

where the constants r_x, r_y, M_x, and M_y are the same as the from the single species experiments. Use the data above to find the best initial conditions and parameters, X_0, Y_0, a_3, and b_3 to fit this model. Also, give the sum of square errors between the data and the model.

c. Find all equilibria for this competition model. Give a formula for the Jacobian matrix for the linearized system. Determine the eigenvalues and associated eigenvectors at each of the equilibria, then discuss the stability of these equilibria. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral).

d. Sketch a phase portrait of this model. Show the representative trajectory in the phase portrait starting from (X_0, Y_0). Use arrows to show the directions of the solutions near each of the equilibria. What does this suggest happens over the long period of time according to this model?