Fall 2016 Math 541

Homework – Solving $f(x) = 0$ Due Tues. 10/11/16

Be sure to include all MatLab programs used to obtain answers.

1. a. Using Newton's method, explain why the sequence

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}},\ n \geq 1,\ x_0 > 0,$$

converges to $\sqrt{A}$. (Hint: You need to think of a function whose root is $\sqrt{A}$.)

b. Write a MatLab program for any A, then use it to find $\sqrt{7}$ to a tolerance of 10^{-10}.

c. If a sequence $\{x_n\} \to x_*$, then the order of convergence of this sequence is defined by:

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^\alpha} = L$$

for constants α and L, where α is the order of convergence. For "large" enough n, the Cauchy approximation defined by

$$\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|^\alpha} \approx L$$

often provides a reasonable estimate to α and L. Use MatLab on a Cauchy approximation to estimate the values of α and L and give those values. Write an explanation of how you did your computation.

2. The previous problem showed how examining the Cauchy sequence from our Newton's method estimates the order of convergence. This problem gives a graphical means of estimating the order of convergence. We also explore the basins of attraction for different roots.

a. For

$$f(x) = x\cos(x) - \sin^2(x),$$

find all the roots on the interval $[0, 5]$ using Newton's method with a tolerance of 10^{-10}.

b. With the rate of convergence given by

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - x_*|^\alpha} = \lambda,$$

then for n large

$$\ln|x_{n+1} - x_*| \sim \alpha \ln|x_n - x_*| + \ln\lambda$$

Use the roots found in Part a for x_*. Store the Newton iterates $\{x_n\}$ in an array and create a plot of $X = \ln|x_n - x_*|$ and $Y = \ln|x_{n+1} - x_*|$. You can use the MatLab program `polyfit(X,Y,1)` to find the best slope and intercept through your data. (Later we'll explain this program in more detail.) Create graphs for each of your roots. Provide a well-labeled graph showing the data points and best fitting line. With this graphical support explain the rate of convergence for each fixed point. Does your rate change based on your initial guess? Again, please elaborate and show plots to make your case.

c. **Basins of Attraction:** We noted in class that it is hard to determine how far away from a root an initial estimate x_0 can be in order to converge to the root x_*. Take our interval $I = [0, 5]$ and select initial points at every 0.1, $x_0 = 0, 0.1, 0.2, ..., 4.9, 5$. Create a program to see where the Newton iterates for each of these x_0 values converge in less than `Nmax = 10` iterates with `tol = 1e-10`. Determine the intervals where the x_0 values converge to the closest x_*. Note when the convergence is to another root and not the closest x_*. Write a paragraph about what you observe from the results of your program and give some explanations for your observations. (Hint: You may want to graph the original function and use the geometric interpretation of Newton's method to help explain your observations.)

WeBWorK: Programs for the numerical methods and discussion about the order of convergence must be included in your written HW. There are **4** WeBWorK problems.