Introduction

We need a proper definition for the integral
Riemann sums provide the basis for the integral
The integral represents the area under a curve
The proper definition suggests means to numerically compute the integral

Salton Sea

Salton Sea: One of the world’s largest inland seas created by accident when a dike broke during the construction of the All-American Canal in 1905

- Popular recreation area for boating and fishing
- Crucial region for birds on migration because loss of water habitat
- Sea is 228 ft below sea level, so water only lost by evaporation
- Agricultural activities result in serious pollution problems
Area of Salton Sea: How can we determine the area of the Salton sea?

- One technique is to cut out the image of the lake and weigh it against a standard measured area.
- Computers have advanced software that measure the area quite accurately by a simple scanning or tracing process.
- Place a refined grid on the picture and determine the area.
- All these schemes use the process of integration.

Area of Salton Sea: Use a gridding scheme over an image.

- The area is determined by counting the number of squares that include the image of the Salton Sea.
  - If a box is at least 50% full, we will count it.
  - If a box is less than 50% full, we will not count it.
- As the boxes get smaller, the estimate of the area of the Salton Sea becomes more accurate.

Area of Salton Sea: Using the 6 mi square grid with 50% rule.

- 8 squares apply to this rule.
- Each square is a 36 square mile area.
- This approximation gives 288 square miles.
- Assuming the actual area of the basin is 360 square miles, the error is 20%.
**Salton Sea** grid with 3 mi on a side

**Area of Salton Sea:** Using the 3 mi square grid with 50% rule

- 33 squares apply to this rule
- Each square is a 9 square mile area
- This approximation gives 297 square miles
- Assuming the actual area of the basin is 360 square miles, the error is 17.5%

**Salton Sea** grid with 1.5 mi on a side

**Area of Salton Sea:** Using the 1.5 mi square grid with 50% rule

- 137 squares apply to this rule
- Each square is a 2.25 square mile area
- This approximation gives 308.25 square miles
- Assuming the actual area of the basin is 360 square miles, the error is 14%
- From the figure it is easy to see that shrinking the squares gives a better and better approximation of the area
**Area under a Curve**

**Area under a Curve:** Consider the function 
\[ f(x) = x^3 - 6x^2 + 9x + 2 \quad \text{for} \quad x \in [0, 5] \]

- The actual area under the curve is **28.75**
- Approximate area with rectangles under the curve
- Divide the interval \( x \in [0, 5] \) into even intervals
- Use the midpoint of the interval to get height of the rectangle
- Examine approximation as intervals get smaller

**Area under a Curve:** Divide \( x \in [0, 5] \) into 5 intervals

\[
A \approx \left( f \left( \frac{1}{2} \right) + f \left( \frac{3}{2} \right) + f \left( \frac{5}{2} \right) + f \left( \frac{7}{2} \right) + f \left( \frac{9}{2} \right) \right) \Delta x = \sum_{i=0}^{4} f \left( i + \frac{1}{2} \right) \cdot 1
\]

This gives
\[
A \approx \sum_{i=0}^{4} \left( \left( i + \frac{1}{2} \right)^3 - 6 \left( i + \frac{1}{2} \right)^2 + 9 \left( i + \frac{1}{2} \right) + 2 \right) = 28.125
\]

This is 2.17% less than the actual area

**Area under a Curve:** Divide \( x \in [0, 5] \) into 10 intervals

\[
A \approx \left( f \left( \frac{1}{10} \right) + f \left( \frac{3}{10} \right) + f \left( \frac{5}{10} \right) + \ldots + f \left( \frac{9}{10} \right) \right) \Delta x = \sum_{i=0}^{9} f \left( i + \frac{1}{10} \right) \cdot 1
\]

This gives
\[
A \approx \sum_{i=0}^{9} \left( \left( i + \frac{1}{10} \right)^3 - 6 \left( i + \frac{1}{10} \right)^2 + 9 \left( i + \frac{1}{10} \right) + 2 \right) = 27.9625
\]

This is 0.68% less than the actual area
Area under a Curve: Height of rectangles from the function
\[ f(x) = x^3 - 6x^2 + 9x + 2 \quad \text{for} \quad x \in [0, 5] \]
- Width of the rectangles are \( \Delta x = \frac{1}{2} \)
- Height of rectangles evaluated at midpoints
- Approximate area satisfies
\[
A \approx \sum_{i=0}^{9} f \left( \frac{i}{2} + \frac{1}{4} \right) \Delta x
\]
- This gives
\[
A \approx \frac{1}{2} \sum_{i=0}^{9} \left( \left( \frac{i}{2} + \frac{1}{4} \right)^3 - 6 \left( \frac{i}{2} + \frac{1}{4} \right)^2 + 9 \left( \frac{i}{2} + \frac{1}{4} \right) + 2 \right) = 28.59375
\]
- This is 0.543% less than the actual area
Introduction

Examples

Riemann Integral

Numerical Methods for Integration

Definition of Riemann Integral

Definition of Riemann Integral

Figures below show a single rectangle in computing area of the Riemann Integral and all of the rectangles using the Midpoint Rule for approximating the area under the curve.
Midpoint Rule for Integration is a method for approximating integrals

- Consider a continuous function \( f(x) \) and an interval \( x \in [a, b] \)
- Subdivide the interval into \( n \) pieces, evaluating the function at the midpoints
- The area under \( f(x) \) is approximated by adding the areas of the rectangles

\[
S_n = \sum_{i=1}^{n} f(c_i) \Delta x
\]

- This is the Midpoint Rule for Integration
- Like Euler’s Method, there are much better numerical methods for integration

Riemann Sums and Riemann Integral

- The Midpoint Rule described above is a specialized form of Riemann sums
- The more general form of Riemann sums allows the subintervals to have varying lengths, \( \Delta x_i \)
- The choice of where the function is evaluated need not be at the midpoint as described above
- The Riemann integral is defined using a limiting process, similar to the one described above

Numerical Methods for Integration

- Many integrals cannot be solved exactly
- The Riemann integral has a number of methods for finding approximate solutions
- The Riemann integral represents the area under a function on a specified interval
- This is a definite integral

\[
\int_{a}^{b} f(x) \, dx
\]
**Midpoint Rule** was discussed above and is reviewed below.

- Let $f(x)$ be a continuous function on the interval $[a, b]$.
- The interval of integration $[a, b]$ is divided into $n$ subintervals $[x_{i-1}, x_i]$ with length $\Delta x = \frac{b-a}{n}$.
- The midpoint of each of these intervals is $c_i = \frac{x_{i-1} + x_i}{2}$.
- Height of an approximating rectangle, $f(c_i)$.
- The **Midpoint Rule** satisfies
  \[
  \int_a^b f(x) \, dx \approx \sum_{i=1}^{n} f(c_i) \Delta x
  \]

**Trapezoid Rule** approximates the area under a curve using trapezoids.

- Let $f(x)$ be a continuous function on the interval $[a, b]$.
- The interval of integration $[a, b]$ is divided into $n$ subintervals $[x_{i-1}, x_i]$ with length $\Delta x = \frac{b-a}{n}$.
- The function is evaluated at the endpoints of the subintervals.
- A line segment is formed between these function evaluations on each subinterval creating a trapezoid.
- The **Trapezoid Rule** satisfies
  \[
  \int_a^b f(x) \, dx \approx \left( \frac{1}{2} f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} f(x_n) \right) \Delta x
  \]

**Diagram for Trapezoid Rule:** Note that the trapezoid rule has a similar accuracy as the **Midpoint Rule**.

**Trapezoid Rule:** Use illustration above

\[
f(x) = x^3 - 6x^2 + 9x + 2 \quad \text{for} \quad x \in [0, 5]
\]

- The interval $[0, 5]$ is divided into 5 subintervals with length $\Delta x = 1$.
- Height of the function are evaluated at endpoints of the subintervals.
- The **Trapezoid Rule** gives
  \[
  \int_0^5 f(x) \, dx \approx \left( \frac{1}{2} f(0) + f(1) + f(2) + f(3) + f(4) + \frac{1}{2} f(5) \right) \Delta x
  \]
  \[
  = \left( \frac{1}{2}2 + 6 + 4 + 2 + 6 + \frac{1}{2}22 \right) \cdot 1 = 30
  \]

- The actual integral value is **28.75**, so the approximation is 4.3% too high (similar error to the midpoint rule).
Simpson’s Rule obtains a much more accurate approximation to the integral without having a significantly more complicated formula.

- Simpson’s rule approximates the function $f(x)$ by quadratics
- The interval of integration $[a, b]$ is divided $n$ subintervals $[x_{i-1}, x_i]
  - Length $\Delta x = \frac{b-a}{n}$
  - The endpoints are $x_0 = a$ and $x_n = b$
  - $n$ must be an even integer
- The formula for Simpson’s rule is

\[
\int_a^b f(x) \, dx \approx \left( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right) \frac{\Delta x}{3}
\]

Example: Use the Midpoint rule, Trapezoid rule, and Simpson’s rule to approximate the integral

\[
\int_0^2 x^2 \, dx
\]

with $n = 4$

Solution: With $n = 4$ the four subintervals are $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, \frac{3}{2}]$, and $[\frac{3}{2}, 2]$, so $\Delta x = \frac{1}{2}$

The midpoints are $c_i = \frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, and $\frac{7}{4}$

Solution: With $\Delta x = \frac{1}{2}$, the Midpoint rule gives

\[
\int_0^2 x^2 \, dx \approx \sum_{i=1}^4 f(c_i) \Delta x
\]

\[
= \sum_{i=1}^4 \left( \frac{1}{2} - \frac{1}{4} \right)^2 \frac{1}{2} \frac{1}{2}
\]

\[
= \left( \frac{1+9+25+49}{16} \right) \frac{1}{2}
\]

\[
= \frac{21}{8} = 2.625
\]

Solution: With $\Delta x = \frac{1}{2}$, the Trapezoid rule gives

\[
\int_0^2 x^2 \, dx \approx \left( \frac{1}{2} f(x_0) + \sum_{i=1}^3 f(x_i) + \frac{1}{2} f(x_4) \right) \Delta x
\]

\[
= \left( \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right) + (1)^2 + \left( \frac{3}{2} \right)^2 + \frac{1}{2} (2) \right) \frac{1}{2}
\]

\[
= 2.75
\]
**Example 2:** Consider the function
\[ f(x) = 9 - x^2 \]
- Find the area in the first quadrant under the curve
- Sketch a graph showing the area under the graph
- Use the **Midpoint rule**, **Trapezoid rule**, and **Simpson’s rule** to approximate the integral with \( n = 6 \)

**Solution:** The function intersects the \( x \)-axis at \( x = 3 \)

**Solution (cont):** The integral defining the area in the previous figure is
\[ \int_0^3 (9 - x^2) \, dx \]
- The integral has limits \( x = 0 \) and \( x = 3 \), so with \( n = 6 \) the subintervals have length, \( \Delta x = \frac{1}{2} \)
- The midpoints of the subintervals are\[ c_i = \frac{i}{2} - \frac{1}{4} \quad i = 1, \ldots, 6 \]

**Solution:** With \( \Delta x = \frac{1}{2} \), **Simpson’s rule** gives
\[
\int_0^2 x^2 \, dx \approx \left(f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + f(2)\right) \frac{\Delta x}{3}
\]
\[
= \left(0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right) \frac{1}{6}
\]
\[
= \frac{8}{3}
\]
This is the exact answer. Simpson’s rule gives the exact answer for any quadratic.
Example 2

**Solution (cont):** With $\Delta x = \frac{1}{2}$, the **Midpoint rule** gives

$$
\int_{0}^{3} (9 - x^2)dx \approx \sum_{i=1}^{6} f(c_i) \Delta x
$$

$$
= \sum_{i=1}^{6} \left( 9 - \left( \frac{i}{2} - \frac{1}{4} \right)^2 \right) \frac{1}{2}
$$

$$
= (8.9375 + 8.4375 + 7.4375 + 5.9375 + 3.9375 + 1.4375) \frac{1}{2}
$$

$$
= 18.0625
$$

-----

Example 2

**Solution:** With $\Delta x = \frac{1}{2}$ and $x_i = \frac{i}{2}$, the **Trapezoid rule** gives

$$
\int_{0}^{2} (9 - x^2)dx \approx \left( \frac{1}{2} f(0) + \sum_{i=1}^{5} f(x_i) + \frac{1}{2} f(3) \right) \Delta x
$$

$$
= (4.5 + 8.75 + 8 + 6.75 + 5 + 2.75 + 0) \frac{1}{2}
$$

$$
= 17.875
$$

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Example 2

**Temperature Example:** Insects are an important agricultural pest

- Some pesticides have their greatest effects at particular stages of the insect development
- Timing of application of the pesticide can be very significant
- Maturation of insects is often dependent upon temperature more than length of time
- It can be important to track the cumulative temperature rather than the length of time that an insect has been around
- Cumulative temperature $T_c$ (in °C-hr) is found by integrating the temperature $T(t)$ over a period of time

$$
T_c = \int_{a}^{b} T(t) dt
$$
Temperature Example: Data for temperatures (noon to 7 PM)

<table>
<thead>
<tr>
<th>Time</th>
<th>12:00</th>
<th>13:00</th>
<th>14:00</th>
<th>15:00</th>
<th>16:00</th>
<th>17:00</th>
<th>18:00</th>
<th>19:00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp(°C)</td>
<td>33</td>
<td>34</td>
<td>36</td>
<td>35</td>
<td>32</td>
<td>30</td>
<td>26</td>
<td>24</td>
</tr>
</tbody>
</table>

Use the Trapezoid rule and the data from the table to approximate the cumulative temperature from noon to 7 PM

**Note:** The average temperature is 31.25 °C

**Solution:** Since the length of time between the temperature measurements is one hour, $\Delta t = 1$

The Trapezoid rule gives

$$T_c = \int_{12}^{19} T(t)dt \approx \left( \frac{1}{2} T(12) + \sum_{i=13}^{18} T(i) + \frac{1}{2} T(19) \right) \Delta t$$

$$= (16.5 + 34 + 36 + 35 + 32 + 30 + 26 + 12) \cdot 1$$

$$= 221.5 \, ^{\circ}\text{C} \cdot \text{hr}$$

This varies slightly from computing the average temperature and multiplying by the length of time ($31.25 \times 7 = 218.75$)