Modeling and Differential Equations

- **Biological Models** often use **differential equations**, which have no analytical solution
- Properties of the model and differential equation can provide some insight
- **Qualitative analysis** techniques provide simple tools for understanding models
- Analysis helps understand equilibria and behavior near the equilibria

Growth of Bacteria

- Bacterial growth usually follows a regular pattern
  - They are inoculated into a broth
  - Culture has a **lag period** (adjusting to the new growing conditions)
  - A period of **exponential growth** (Malthusian growth model)
  - Cell growth slows to **stationary growth** (nutrients become limiting or waste products build)
  - Population levels off or declines using different pathways to survive the lean times
Bacterial Growth Experiments

Bacterial Growth Experiments

- *Staphylococcus aureus* is a common pathogen that can cause food poisoning
- Cultures of this bacterium satisfy the **logistic growth**
  - Data from one experiment by Carl Gunderson (Lab of Anca Segall)
  - Normal strain is grown to the optical density (OD\(_{650}\)), which estimates the number of bacteria

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Malthusian Growth

Malthusian Growth

- The experiment shows the rapid initial growth of the bacteria
  - This suggests **Malthusian growth**
- Earlier showed the **Malthusian growth model**:
  \[
  \frac{dP}{dt} = rP, \quad P(0) = P_0
  \]
  - Solution satisfies:
  \[
  P(t) = P_0e^{rt}
  \]
  - Only later does crowding require **logistic growth model**
Malthusian Growth

Graph shows early data with Malthusian and logistic growth models.
Best fitting Malthusian growth model is
\[ P(t) = 0.0279 e^{0.905 t} \]

Logistic Growth

- After rapid growth the experiment shows bacterial growth slowing.
- This suggests Logistic growth.
- Earlier showed the Logistic growth model:
  \[ \frac{dP}{dt} = rP \left(1 - \frac{P}{M}\right), \quad P(0) = P_0 \]
- Solution satisfies:
  \[ P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}} \]
- This solution is found by:
  - Separation of Variables and special integration techniques
  - Bernoulli’s method
- Want easier methods to investigate qualitative behavior.

Qualitative Behavior of Differential Equations

- Techniques follow analysis of discrete dynamical systems.
  - Find the equilibria.
  - Determine the behavior of the solutions near the equilibria.
- Study Autonomous Differential Equations
  \[ \frac{dy}{dt} = f(y) \]
  - Graph function \( f(y) \).
  - Find Phase Portrait and determine local behavior near equilibria.
Example: Logistic Growth Model
Consider the logistic growth equation:
\[
\frac{dP}{dt} = f(P) = 0.05P \left( 1 - \frac{P}{2000} \right)
\]

- The graph above of \( f(P) \) shows
  - \( f(P) \) intersects the \( P \)-axis at \( P = 0 \) and \( P = 2000 \)
  - These \( P \)-intercepts are where \( f(P) = 0 \) or \( \frac{dP}{dt} = 0 \)

- There is no change in the growth of the population or the population is at equilibrium
- This is the first step in any qualitative analysis: Find all equilibria \( f(P) = 0 \)

Local Analysis: **Equilibria** are \( P_e = 0 \) and \( P_e = 2000 \)
- The graph of \( f(P) \) gives more information
- To the left of \( P_e = 0 \), \( f(P) < 0 \)
  - Since \( \frac{dP}{dt} = f(P) < 0 \), \( P(t) \) is decreasing
  - Note that this region is outside the region of biological significance
- For \( 0 < P < 2000 \), \( f(P) > 0 \)
  - Since \( \frac{dP}{dt} = f(P) > 0 \), \( P(t) \) is increasing
  - Population monotonically growing in this area
- For \( P > 2000 \), \( f(P) < 0 \)
  - Since \( \frac{dP}{dt} = f(P) < 0 \), \( P(t) \) is decreasing
  - Population monotonically decreasing in this region

Phase Portrait
- Use the above information to draw a Phase Portrait of the behavior of this differential equation along the \( P \)-axis
- The behavior of the differential equation is denoted by arrows along the \( P \)-axis
  - When \( f(P) < 0 \), \( P(t) \) is decreasing and we draw an arrow to the left
  - When \( f(P) > 0 \), \( P(t) \) is increasing and we draw an arrow to the right
- **Equilibria**
  - A solid dot represents an equilibrium that solutions approach or stable equilibrium
  - An open dot represents an equilibrium that solutions go away from or unstable equilibrium
Phase Portrait

Summary of Qualitative Analysis

- Graph shows solutions either moving away from the equilibrium at $P_e = 0$ or moving toward $P_e = 2000$
- Solutions are increasing most rapidly where $f(P)$ is at a maximum
- Phase portrait shows direction of flow of the solutions without solving the differential equation
- Solutions cannot cross in the $tP$-plane
- **Phase Portrait analysis**
  - Behavior of a scalar differential equation found by just graphing function
  - **Equilibria** are zeros of function
  - Direction of flow/arrows from sign of function
  - **Stability** of equilibria from whether arrows point toward or away from the equilibria

Example: Sine Function

Consider the differential equation:

$$\frac{dx}{dt} = 2 \sin(\pi x)$$

- Find all equilibria
- Determine the stability of the equilibria
- Sketch the phase portrait
- Show typical solutions
Example: Sine Function

For the sine function below:
\[
\frac{dx}{dt} = 2 \sin(\pi x)
\]
- The equilibria satisfy
  \[2 \sin(\pi x_e) = 0\]
- Thus, \(x_e = n\), where \(n\) is any integer
- The sine function passes from negative to positive through \(x_e = 0\), so solutions move away from this equilibrium
- The sine function passes from positive to negative through \(x_e = 1\), so solutions move toward this equilibrium
- From the function behavior near equilibria
  - All equilibria of the form \(x_e = 2n\) (even integer) are **unstable**
  - All equilibria of the form \(x_e = 2n + 1\) (odd integer) are **stable**

Phase Portrait: Since \(2 \sin(\pi x)\) alternates sign between integers, the phase portrait follows below:

Diagram of Solutions for Sine Model

- The shell of a snail exhibits **chirality**, left-handed (sinistral) or right-handed (dextral) coil relative to the central axis
- The Indian conch shell, *Turbinella pyrum*, is primarily a right-handed gastropod [1]
- The left-handed shells are “exceedingly rare”
- The Indians view the rare shells as very holy
  - The Hindu god “Vishnu, in the form of his most celebrated avatar, Krishna, blows this sacred conch shell to call the army of Arjuna into battle”
- So why does nature favor snails with one particular handedness?
- Gould notes that the vast majority of snails grow the dextral form.

Clifford Henry Taubes [2] gives a simple mathematical model to predict the bias of either the dextral or sinistral forms for a given species.

Assume that the probability of a dextral snail breeding with a sinistral snail is proportional to the product of the number of dextral snails times sinistral snails.

Assume that two sinistral snails always produce a sinistral snail and two dextral snails produce a dextral snail.

Assume that a dextral-sinistral pair produce dextral and sinistral offspring with equal probability.

By the first assumption, a dextral snail is twice as likely to choose a dextral snail than a sinistral snail.

Could use real experimental verification of the assumptions

Figure 3: Diagram of Solutions for Snail Model

Snail Model - Summary
- Figures show the solutions tend toward one of the stable equilibria, \( p_e = 0 \) or \( 1 \)
- When the solution tends toward \( p_e = 0 \), then the probability of a dextral snail being found drops to zero, so the population of snails all have the sinistral form
- When the solution tends toward \( p_e = 1 \), then the population of snails virtually all have the dextral form
- This is what is observed in nature suggesting that this model exhibits the behavior of the evolution of snails
- This does not mean that the model is a good model!
- It simply means that the model exhibits the basic behavior observed experimentally from the biological experiments

Pitchfork Bifurcation

- **Bifurcations** are when behavior of the differential equation changes
- A supercritical pitchfork bifurcation has differing numbers of equilibria as a parameter changes
- Consider the differential equation:
  \[
  \frac{dx}{dt} = \alpha x - x^3,
  \]
  where \( \alpha \) can be positive, negative, or zero
- Find all equilibria
- Determine the stability of those equilibria as \( \alpha \) changes
- For \( \alpha = \pm 1 \), sketch the phase portraits and show typical solutions in the \( x \) and \( t \) solution space

For the differential equation:

\[
\frac{dx}{dt} = \alpha x - x^3
\]

- Find equilibria by solving
  \[
  \alpha x_e - x_e^3 = 0 \quad \text{or} \quad x_e(\alpha - x_e^2) = 0
  \]
- There is always an equilibrium at \( x_e = 0 \)
- If \( \alpha < 0 \), then \( x_e = 0 \) is the only equilibrium
- If \( \alpha > 0 \), then there are three equilibria given by
  \[
  x_e = 0, \pm \sqrt{\alpha}
  \]
For the differential equation:
\[
\frac{dx}{dt} = \alpha x - x^3
\]

- When \( x_e = 0 \) is the only equilibrium, then it is **stable**
- When there are three equilibria, then \( x_e = 0 \) is an **unstable** equilibrium, while the equilibria, \( x_e = \pm \sqrt{\alpha} \), are both **stable**
- As the parameter \( \alpha \) changes from negative to positive, the differential equation’s qualitative behavior changes
  - From having a single stable equilibrium at \( x_e = 0 \)
  - To three equilibria with \( x_e = 0 \) becoming unstable and the other two being stable

**Phase Portrait:** For \( \alpha = -1 \),
\[
\frac{dx}{dt} = -x - x^3
\]
The function is always decreasing, intersecting the \( x \)-axis at \( x_e = 0 \)

Diagram of Solutions: For \( \alpha = -1 \) with \( \frac{dx}{dt} = -x - x^3 \)

**Phase Portrait:** For \( \alpha = 1 \),
\[
\frac{dx}{dt} = x - x^3
\]
The function intersects the \( x \)-axis at \( x_e = 0, \pm 1 \)
**Pitchfork Bifurcation**

**Diagram of Solutions:** For $\alpha = 1$ with $\frac{dx}{dt} = x - x^3$

\[ \alpha = +1 \]

**Allee Effect**

**Thick-Billed Parrot:** *Rhynchopsitta pachyrhyncha*

- A gregarious montane bird that feeds largely on conifer seeds, using its large beak to break open pine cones for the seeds.
- These birds used to fly in huge flocks in the mountainous regions of Mexico and Southwestern U.S.
- Largely because of habitat loss, these birds have lost much of their original range and have dropped to only about 1500 breeding pairs in a few large colonies in the mountains of Mexico.
- The pressures to log their habitat puts this population at extreme risk for extinction.

**Thick-Billed Parrot:** *Rhynchopsitta pachyrhyncha*

- The populations of these birds appear to exhibit a property known in ecology as the **Allee effect**
- These parrots congregate in large social groups for almost all of their activities.
- The large group allows the birds many more eyes to watch out for predators.
- When the population drops below a certain number, then these birds become easy targets for predators, primarily hawks, which adversely affects their ability to sustain a breeding colony.
**Allee Effect:**
- Suppose that a population study on thick-billed parrots in a particular region finds that the population, $N(t)$, of the parrots satisfies the differential equation:
  \[
  \frac{dN}{dt} = N \left( r - a(N - b)^2 \right),
  \]
  where $r = 0.04$, $a = 10^{-8}$, and $b = 2200$
- Find the equilibria for this differential equation
- Determine the stability of the equilibria
- Draw a phase portrait for the behavior of this model
- Describe what happens to various starting populations of the parrots as predicted by this model

**Equilibria:**
- Set the right side of the differential equation equal to zero:
  \[N_e \left( r - a(N_e - b)^2 \right) = 0\]
- One solution is the trivial or extinction equilibrium, $N_e = 0$
- When $(r - a(N_e - b)^2) = 0$, then
  \[(N_e - b)^2 = \frac{r}{a} \quad \text{or} \quad N_e = b \pm \sqrt{\frac{r}{a}}\]
- Three distinct equilibria unless $r = 0$ or $b = \sqrt{r/a}$
- With the parameters $r = 0.04$, $a = 10^{-8}$, and $b = 2200$, the equilibria are
  \[N_e = 0 \quad N_e = 200 \quad 4200\]

**Phase Portrait:** Graph of right hand side of differential equation showing equilibria and their stability

**Solutions:** For
\[
\frac{dN}{dt} = N \left( r - a(N - b)^2 \right)
\]
**Interpretation: Model of Allee Effect**

- From the phase portrait, the equilibria at 4200 and 0 are stable.
- The *threshold* equilibrium at 200 is unstable.
  - If the population is above 200, then it goes to the *carrying capacity* of this region and reaches the stable population of 4200.
  - If the population falls below 200, then this model predicts *extinction*, \( N_c = 0 \).
- This agrees with the description for these social birds, which require a critical number of birds to avoid predation.
- Below this critical number, the predation increases above reproduction, and the population of parrots goes to extinction.
- If the parrot population is larger than 4200, then their numbers will be reduced by starvation (and predation) to the carrying capacity, \( N_c = 4200 \).