Introduction

Animals are frequently devising optimal strategies to gain advantage
- Reproducing more rapidly
- Better protection from predation

Primitive animals long ago split into the prokaryotes (bacterial cells) and eukaryotes (cells in higher organisms like yeast or humans) from a common ancestor
- One argument contends that eukaryotic cells added complexity, size, and organization for advantage in competition
- Prokaryotes stripped down their genome (eliminated junk DNA) to the minimum required for survival, to maximize reproduction

These arguments suggest that organisms try to optimize their situation to gain an advantage

Crow Predation on Whelks

Sea gulls and crows have learned to feed on various mollusks by dropping their prey on rocks to break the protective shells
**Optimal Foraging** – Northwestern crows (*Corvus caurinus*) on Mandarte Island

- Reto Zach studied Northwestern crows on Mandarte Island, British Columbia to learn about foraging for whelks (*Thais lamellosa*). Ecologists study these behaviors to give insight into optimal foraging.
- Northwestern crows:
  - Perch above beaches, then fly to intertidal zone
  - Select largest whelks
  - Fly to the rocky area and drop whelks
  - Eat broken whelks

**Foraging Strategy**

- Whelk Selection
  - Crows search intertidal zone for largest whelks
  - Take whelks to a favorite rocky area

- Flight Strategy
  - Fly to height of about 5 meters
  - Drop whelks on rocks, repeatedly averaging 4 times
  - Eat edible parts when split open

- Can this behavior be explained by an optimal foraging decision process?
- Is the crow exhibiting a behavior that minimizes its expenditure of energy to feed on whelks?

**Why large whelks?**

- Zach experiment:
  - Collected and sorted whelks by size
  - Dropped whelks from various heights until they broke
  - Recorded how many drops at each height were required to break each whelk
Large Whelks

- Easier to break open larger whelks, so crows selectively chose the largest available whelks
- There was a gradient of whelk size on the beach, suggesting that the crows’ foraging behavior was affecting the distribution of whelks in the intertidal zone, with larger whelks further out
- Crows benefit by selecting the larger ones because they don’t need as many drops per whelk, and they gain more energy from consuming a larger one
- Study showed that the whelks broken on the rocks were remarkably similar in size, weighing about 9 grams

Zach Observation – Height of the drops and number of drops required for many crows to eat whelks used a marked pole on the beach near a favorite dropping location

Mathematical Model for Energy

- Energy is directly proportional to the vertical height that an object is lifted (Work put into a system)
- The energy that a crow expends breaking open a whelk
  - The amount of time the crow uses to search for an appropriate whelk
  - The energy in flying to the site where the rocks are
  - The energy required to lift the whelk to a certain height and drop it times the number of vertical flights required to split open the whelk
- Concentrate only on this last component of the problem, as it was observed that the crows kept with the same whelk until they broke it open rather than searching for another whelk when one failed to break after a few attempts
**Energy Function**

- The energy is given by the height \( H \) times the number of drops \( N \) or
  \[
  E = kHN
  \]
  where \( k \) is a constant of proportionality
- Flying higher and increasing the number of drops both increase the use of energy

**Fitting the Data** – Zach’s data on dropping large whelks

<table>
<thead>
<tr>
<th>( H(m) )</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(H) )</td>
<td>56</td>
<td>20</td>
<td>10.2</td>
<td>7.6</td>
<td>6</td>
<td>5</td>
<td>4.3</td>
<td>3.8</td>
<td>3.1</td>
<td>2.5</td>
</tr>
</tbody>
</table>

- Since it always requires at least one drop, the proposed function for the number of drops, \( N \), as a function of height, \( H \) is
  \[
  N(H) = 1 + \frac{a}{H - b}
  \]
- The least squares best fit of this function to Zach’s data gives \( a = 15.97 \) and \( b = 1.209 \)

**Graph for Whelks being Dropped**

**Graph of Energy Function** – The energy function is

\[
E(H) = kH \left( 1 + \frac{a}{H - b} \right)
\]
Mathematical Model for Energy

**Minimization Problem** – Energy satisfies

\[ E(H) = kH \left(1 + \frac{a}{H - b}\right) \]

- A **minimum energy** is apparent from the graph with the value around 5.6 m, which is close to the observed value that Zach found the crows to fly when dropping whelks.
- The derivative of \( E(H) \) is

\[ E'(H) = k \left(1 + \frac{a}{H - b} - \frac{aH}{(H - b)^2}\right) = k \left(\frac{H^2 - 2bH + b^2 - ab}{(H - b)^2}\right) \]

- The **optimal energy** occurs at the **minimum**, where \( E'(H) = 0 \)

**Optimal Solution**

- One application of the derivative is to find critical points where often a function has a **relative minimum** or **maximum**.
- An **optimal solution** for a function is when the function takes on an **absolute minimum** or **maximum** over its domain.

**Definition:** An **absolute minimum** for a function \( f(x) \) occurs at a point \( x = c \), if \( f(c) < f(x) \) for all \( x \) in the domain of \( f \).
Absolute Extrema of a Polynomial: Consider the cubic polynomial $f(x)$ defined on the interval $x \in [0, 5]$, where

$$f(x) = x^3 - 6x^2 + 9x + 4$$

Find the **absolute extrema** of this polynomial on its **domain**

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**Solution:**

The cubic polynomial $f(x) = x^3 - 6x^2 + 9x + 4$

- The derivative is $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$
- Critical points occur at $x_c = 1$ and $x_c = 3$
- To find the **absolute extrema**, we evaluate $f(x)$ at the **critical points** and the **endpoints** of the domain
  - $f(0) = 4$ (an **absolute minimum**)
  - $f(1) = 8$ (an **relative maximum**)
  - $f(3) = 4$ (an **absolute minimum**)
  - $f(5) = 24$ (an **absolute maximum**)

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**Optimal Study Area:** An ecology student goes into the field with 120 m of string and wants to create two adjacent rectangular study areas with the maximum area possible.
Solution – Optimal Study Area: The Objective Function for this problem is the area of the rectangular plots. The area of each rectangular plot is

\[ A(x, y) = xy \]

The optimal solution uses all string. The Constraint Condition is the length of string available

\[ P(x, y) = 4x + 3y = 120 \]

Solution (cont): This problem allows the objective function of two variables to be reduced by the constraint condition to a function of one variable that can readily be optimized.

- The constraint condition is solved for \( y \) to give

\[ y = \frac{120 - 4x}{3} \]

- The objective function becomes

\[ A(x) = x \frac{120 - 4x}{3} = 40x - \frac{4x^2}{3} \]

- The domain of this function is \( x \in [0, 30] \)

Solution (cont): The objective function is a parabola. The optimal solution is the maximum area for the function

\[ A(x) = 40x - \frac{4x^2}{3} \]

- The maximum area occurs at the vertex of this parabola.
- Alternately, we differentiate the objective function with

\[ A'(x) = 40 - \frac{8x}{3} \]

- The critical point occurs when \( A'(x_c) = 0 \) or \( x_c = 15 \).
- The maximum area occurs with \( x = 15 \) m and \( y = 20 \) m.
- To maximize the study areas, the ecology student should make each of the two study areas 15 m wide and 20 m long or \( A_{\text{max}} = 300 \text{ m}^2 \).
Chemical Reaction: One of the simplest chemical reactions is the combination of two substances to form a third

\[ A + B \xrightarrow{k} X \]

- Assume the initial concentration of substance \( A \) is \( a \) and the initial concentration of \( B \) is \( b \)
- The law of mass action gives the following reaction rate
  \[ R(x) = k(a - x)(b - x), \quad 0 \leq x \leq \min(a, b) \]
- \( k \) is the rate constant of the reaction and \( x \) is the concentration of \( X \) during the reaction
- What is the concentration of \( X \) where the reaction rate is at a maximum?

Chemical Reaction 1

Chemical Reaction: Suppose that \( k = 50 \) (sec\(^{-1}\)), \( a = 6 \) (ppm), and \( b = 2 \) (ppm), so

\[ R(x) = 50(6 - x)(2 - x) = 50x^2 - 400x + 600, \quad 0 \leq x \leq 2 \]

- The derivative is
  \[ R'(x) = 100x - 400 \]
- The critical point (where \( R'(x) = 0 \)) is \( x_c = 4 \)
- This critical point is outside the domain (and produces a negative reaction rate)
- At the endpoints
  - At \( x = 0 \), the reaction rate is \( R(0) = 600 \) (maximum)
  - At \( x = 2 \), the reaction rate is \( R(2) = 0 \) (minimum)

Chemical Reaction 2

Chemical Reaction: Graphing the Reaction Rate

Chemical Reaction 3

Example - Nectar Foraging by Bumblebees:\(^1\): Animals often forage on resources that are in discrete patches

- Bumblebees forage on many flowers
- The amount of nectar, \( N(t) \), consumed increases with diminishing returns with time \( t \)
- Suppose this function satisfies
  \[ N(t) = \frac{0.3t}{t + 2}, \]
  with \( t \) in sec and \( N \) in mg
- Assume the travel time between flowers is 4 sec

Nectar Foraging: Assume bumblebees acquire nectar according to

\[ N(t) = \frac{0.3t}{t+2}, \]

with travel between flowers being 4 sec.

- If the bee spends \( t \) sec at each flower, then create a foraging function, \( f(t) \), describing the nectar consumed over one cycle from landing on one flower until landing on the next flower.
- Find the optimal foraging time for receiving the maximum energy gain from the nectar.

Optimal Nectar Foraging: Since

\[ f(t) = \frac{N(t)}{t+4} = \frac{0.3t}{(t+2)(t+4)} = \frac{0.3t}{t^2 + 6t + 8}, \]

the derivative satisfies

\[ f'(t) = \frac{0.3(t^2 + 6t + 8) - 0.3(2t + 6)}{(t^2 + 6t + 8)^2} = \frac{-0.3t^2 + 2.4}{(t^2 + 6t + 8)^2}. \]

The maximum of \( f(t) \) is when \( f'(t) = 0 \), which occurs when

\[-0.3t^2 + 2.4 = 0 \quad \text{or} \quad t = 2\sqrt{2} \approx 2.83 \text{ sec}.\]
**Wire Problem**

**Solution:** The circle has area $\pi r^2$, and the square has area $x^2$.

The **Objective Function** to be optimized is

$$A(r, x) = \pi r^2 + x^2$$

The **Constraint Condition** based on the length of the wire

$$L = 2\pi r + 4x$$

with domain $x \in [0, \frac{L}{4}]$.

From the constraint, $r$ satisfies

$$r = \frac{L - 4x}{2\pi}$$

**Solution:** With the constraint condition, the area function becomes

$$A(x) = \frac{(L - 4x)^2}{4\pi} + x^2$$

Differentiating $A(x)$ gives

$$A'(x) = \frac{2(L - 4x)(-4)}{4\pi} + 2x = 2\left(\frac{4 + \pi}{\pi}\right)x - \frac{L}{\pi}$$

**Relative extrema** satisfy $A'(x) = 0$, so

$$(4 + \pi)x = L$$

**Solution:** The relative extremum occurs at

$$x = \frac{L}{4 + \pi}$$

- The second derivative of $A(x)$ is
  $$A''(x) = \frac{8}{\pi} + 2 > 0$$

- The function is concave upward, so the **critical point** is a **minimum**.

- $A(x)$ is a quadratic with the leading coefficient being positive, so the vertex of the parabola is the minimum.

- Cutting the wire at $x = \frac{L}{4 + \pi}$ gives the minimum possible area.

**Solution: To find the maximum the Theorem for an Optimal Solution requires checking the endpoints**

- The **endpoints**
  - All in the circle, $x = 0$, $A(0) = \frac{L^2}{4\pi}$
  - All in the square, $x = \frac{L}{4}$, $A\left(\frac{L}{4}\right) = \frac{L^2}{16}$

- Since $4\pi < 16$, $A(0) > A\left(\frac{L}{4}\right)$

- The **maximum** occurs when the wire is used to create a circle.

- Geometrically, a circle is the most efficient conversion of a linear measurement into area.
Blood Vessel Branching

A smaller blood vessel, radius $r_2$, is shown branching off a primary blood vessel, radius $r_1$ with angle $\theta$.

Our goal is to find the optimal angle of branching that minimizes the energy required to transport the blood.

The primary loss of energy for flowing blood is the resistance in the blood vessels.

Resistance follows Poiseuille’s Law

$$ R = C \frac{L}{r^4}, $$

where $C$ is a constant, $L$ is the length of the vessel, and $r$ is the radius of the blood vessel.

Problem begins by finding the lengths $|AB|$ and $|BC|$ as a function of $\theta$ with respect to the fixed distances $a$ and $b$.

From our basic definitions of trig functions we have

$$ \sin(\theta) = \frac{b}{|BC|} \quad \text{or} \quad |BC| = \frac{b}{\sin(\theta)} $$

Also,

$$ \cos(\theta) = \frac{a - |AB|}{|BC|} = \frac{(a - |AB|) \sin(\theta)}{b} \quad \text{or} \quad |AB| = a - \frac{b \cos(\theta)}{\sin(\theta)} $$

Apply Poiseuille’s Law using the lengths $|AB|$ and $|BC|$ to obtain the resistance $R(\theta)$

$$ R(\theta) = C \left( \frac{a}{r_1^4} - \frac{b \cos(\theta)}{r_1^4 \sin(\theta)} + \frac{b}{r_2^4 \sin(\theta)} \right) $$

Differentiating $R(\theta)$ gives

$$ R'(\theta) = Cb \left[ -\frac{1}{r_1^4} \left( -\sin(\theta) \sin(\theta) - \cos(\theta) \cos(\theta) \right) - \frac{\cos(\theta)}{r_2^4 \sin^2(\theta)} \right] $$

$$ R'(\theta) = \frac{Cb}{\sin^2(\theta)} \left( \frac{1}{r_1^4} - \frac{\cos(\theta)}{r_2^4} \right) $$
Since
\[ R'(\theta) = \frac{C_b}{\sin^2(\theta)} \left( \frac{1}{r_1^2} - \frac{\cos(\theta)}{r_2^2} \right), \]

the **minimum resistance** occurs when \( R'(\theta) = 0 \) or
\[ \cos(\theta) = \frac{r_2^4}{r_1^4} \quad \text{or} \quad \theta = \arccos \left( \frac{r_2^4}{r_1^4} \right) \]

For small ratios of \( \frac{r_2}{r_1} \), the new blood vessel optimally comes off at nearly a right angle.

When \( \frac{r_2}{r_1} = \frac{1}{2} \), the optimal angle is \( \theta = 1.5083 \approx 86.4^\circ \).

Even when \( \frac{r_2}{r_1} = \frac{3}{4} \), the optimal angle is \( \theta = 1.2489 \approx 71.6^\circ \).