Outline

1. Introduction
   - Pollution in a Lake

2. Euler’s Method
   - Malthusian Growth Example
   - Example with $f(t, y)$
   - Numerical Solution of the Lake Problem
   - More Examples
   - Time-varying Population Model

3. Improved Euler’s Method
   - Example
Introduction

- Differential Equations provide useful models
- Realistic Models are often Complex
- Most differential equations can **not** be solved exactly
- Develop numerical methods to solve differential equations
Pollution in a Lake

- Previously studied a simple model for Lake Pollution
- Complicate by adding time-varying pollution source
- Include periodic flow for seasonal effects
- Present numerical method to simulate the model
Non-point Source of Pollution and Seasonal Flow Variation

Consider a non-point source, such as agricultural runoff of pesticide
- Assume a pesticide is removed from the market
- If the pesticide doesn’t degrade, it leaches into runoff water

Concentration of the pesticide in the river being time-varying
- Typically, there is an exponential decay after the use of the pesticide is stopped
  - Example of concentration

\[ p(t) = 5 \, e^{-0.002t} \]
Including Seasonal Effects

- River flows vary seasonally
- Assume lake maintains a constant volume, $V$
- Seasonal flow (time varying) entering is reflected with same outflowing flow
  - Example of sinusoidal annual flow

$$f(t) = 100 + 50 \cos(0.0172t)$$
Pollution in a Lake

**Mathematical Model:** Use Mass Balance

The change in amount of pollutant = Amount entering - Amount leaving

- Amount entering is concentration of the pollutant in the river times the flow rate of the river
  \[ f(t)p(t) \]

- Amount leaving a well-mixed lake is concentration of the pollutant in the lake times the flow rate of the river
  \[ f(t)c(t) \]

- The amount of pollutant in the lake, \( a(t) \), satisfies
  \[ \frac{da}{dt} = f(t)p(t) - f(t)c(t) \]
Pollution in a Lake

**Mathematical Model:** Let the concentration be \( c(t) = \frac{a(t)}{V} \)

\[
\frac{dc(t)}{dt} = \frac{f(t)}{V} (p(t) - c(t)) \quad \text{with} \quad c(0) = c_0
\]

- Assume that the volume of the lake is 10,000 m\(^3\) and the initial level of pollutant in the lake is \( c_0 = 5 \) ppm
- With \( p(t) \) and \( f(t) \) from before, model is

\[
\frac{dc(t)}{dt} = (0.01 + 0.005 \cos(0.0172t))(5e^{-0.002t} - c(t))
\]

- Complicated, but an exact solution exists
- Show an easier numerical method to approximate the solution
Euler’s Method

Initial Value Problem: Consider

\[
\frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(t_0) = y_0
\]

- From the definition of the derivative

\[
\frac{dy}{dt} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}
\]

- Instead of taking the limit, fix \( h \), so

\[
\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h}
\]

- Substitute into the differential equation and with algebra write

\[
y(t+h) \approx y(t) + hf(t, y)
\]
Euler’s Method for a fixed $h$ is

$$y(t + h) = y(t) + hf(t, y)$$

- Geometrically, Euler’s method looks at the slope of the tangent line
  - The approximate solution follows the tangent line for a time step $h$
  - Repeat this process at each time step to obtain an approximation to the solution
- The accuracy of this method to track the solution depends on the length of the time step, $h$, and the nature of the function $f(t, y)$
- This technique is rarely used as it has very bad convergence properties to the actual solution
Graph of Euler’s Method

Euler’s Method

$y(t)$

$t_0, t_1, t_2, t_3, t_4, t_n, t_{n+1}$

$h$

$(t_0, y_0), (t_1, y_1), (t_2, y_2), (t_3, y_3), (t_4, y_4), (t_n, y_n), (t_{n+1}, y_{n+1})$
Euler’s Method Formula: Euler’s method is just a discrete dynamical system for approximating the solution of a continuous model

- Let $t_{n+1} = t_n + h$
- Define $y_n = y(t_n)$
- The initial condition gives $y(t_0) = y_0$
- **Euler’s Method** is the discrete dynamical system

$$y_{n+1} = y_n + h f(t_n, y_n)$$

- Euler’s Method only needs the initial condition to start and the right hand side of the differential equation (the **slope field**), $f(t, y)$ to obtain the approximate solution
Malthusian Growth Example: Consider the model

\[ \frac{dP}{dt} = 0.2P \quad \text{with} \quad P(0) = 50 \]

Find the exact solution and approximate the solution with Euler’s Method for \( t \in [0, 1] \) with \( h = 0.1 \)

**Solution:** The exact solution is

\[ P(t) = 50e^{0.2t} \]
Solution (cont): The Formula for Euler’s Method is

$$P_{n+1} = P_n + h \cdot 0.2 \cdot P_n$$

The initial condition $P(0) = 50$ implies that $t_0 = 0$ and $P_0 = 50$

Create a table for the Euler iterates

<table>
<thead>
<tr>
<th>$t_n$</th>
<th>$P_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0 = 0$</td>
<td>$P_0 = 50$</td>
</tr>
<tr>
<td>$t_1 = t_0 + h = 0.1$</td>
<td>$P_1 = P_0 + 0.1(0.2P_0) = 50 + 1 = 51$</td>
</tr>
<tr>
<td>$t_2 = t_1 + h = 0.2$</td>
<td>$P_2 = P_1 + 0.1(0.2P_1) = 51 + 1.02 = 52.02$</td>
</tr>
<tr>
<td>$t_3 = t_2 + h = 0.3$</td>
<td>$P_3 = P_2 + 0.1(0.2P_2) = 52.02 + 1.0404 = 53.0604$</td>
</tr>
</tbody>
</table>
Solution (cont): Iterations are easily continued - Below is table of the actual solution and the Euler’s method iterates

<table>
<thead>
<tr>
<th>t</th>
<th>Euler Solution</th>
<th>Actual Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>0.1</td>
<td>51</td>
<td>51.01</td>
</tr>
<tr>
<td>0.2</td>
<td>52.02</td>
<td>52.041</td>
</tr>
<tr>
<td>0.3</td>
<td>53.060</td>
<td>53.092</td>
</tr>
<tr>
<td>0.4</td>
<td>54.122</td>
<td>54.164</td>
</tr>
<tr>
<td>0.5</td>
<td>55.204</td>
<td>55.259</td>
</tr>
<tr>
<td>0.6</td>
<td>56.308</td>
<td>56.375</td>
</tr>
<tr>
<td>0.7</td>
<td>57.434</td>
<td>57.514</td>
</tr>
<tr>
<td>0.8</td>
<td>58.583</td>
<td>58.676</td>
</tr>
<tr>
<td>0.9</td>
<td>59.755</td>
<td>59.861</td>
</tr>
<tr>
<td>1.0</td>
<td>60.950</td>
<td>61.070</td>
</tr>
</tbody>
</table>
Graph of Euler’s Method for Malthusian Growth Example

Euler’s Method – $P' = 0.2 \, P$

**Euler’s Method**

**Actual Solution**

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Lecture Notes – Numerical Methods for Differ
Error Analysis and Larger Stepsize

- The table and the graph shows that Euler’s method is tracking the solution fairly well over the interval of the simulation.
- The error at $t = 1$ is only 0.2%
- However, this is a fairly short period of time and the stepsize is relatively small.
- What happens when the stepsize is increased and the interval of time being considered is larger?
There is a 9% error in the numerical solution at $t = 10$
Euler’s Method with \( f(t, y) \): Consider the model

\[
\frac{dy}{dt} = y + t \quad \text{with} \quad y(0) = 3
\]

Find the approximate solution with Euler’s Method at \( t = 1 \) with stepsize \( h = 0.25 \)

Compare the Euler solution to the exact solution

\[
y(t) = 4e^t - t - 1
\]
Euler’s Method with $f(t, y)$

**Solution:** Verify the actual solution:

1. Initial condition:
   
   $$y(0) = 4e^0 - 0 - 1 = 3$$

2. The differential equation:
   
   $$\frac{dy}{dt} = 4e^t - 1$$
   
   $$y(t) + t = 4e^t - t - 1 + 1 = 4e^t - 1$$

**Euler’s formula** for this problem is

$$y_{n+1} = y_n + h(y_n + t_n)$$
Solution (cont): Euler’s formula with $h = 0.25$ is

$$y_{n+1} = y_n + 0.25(y_n + t_n)$$

<table>
<thead>
<tr>
<th>$t_n$</th>
<th>Euler solution $y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0 = 0$</td>
<td>$y_0 = 3$</td>
</tr>
<tr>
<td>$t_1 = 0.25$</td>
<td>$y_1 = y_0 + h(y_0 + t_0) = 3 + 0.25(3 + 0) = 3.75$</td>
</tr>
<tr>
<td>$t_2 = 0.5$</td>
<td>$y_2 = y_1 + h(y_1 + t_1) = 3.75 + 0.25(3.75 + 0.25) = 4.75$</td>
</tr>
<tr>
<td>$t_3 = 0.75$</td>
<td>$y_3 = y_2 + h(y_2 + t_2) = 4.75 + 0.25(4.75 + 0.5) = 6.0624$</td>
</tr>
<tr>
<td>$t_4 = 1$</td>
<td>$y_4 = y_3 + h(y_3 + t_3) = 6.0624 + 0.25(6.0624 + 0.75) = 7.7656$</td>
</tr>
</tbody>
</table>
Solution (cont): Error Analysis

- $y_4 = 7.7656$ corresponds to the approximate solution of $y(1)$
- The actual solution gives $y(1) = 8.87312$, so the Euler approximation with this large stepsize is not a very good approximation of the actual solution with a 12.5% error
- If the stepsize is reduced to $h = 0.1$, then Euler’s method requires 10 steps to find an approximate solution for $y(1)$
- It can be shown that the Euler approximate of $y(1)$, $y_{10} = 8.37497$, which is better, but still has a 5.6% error
Numerical Solution of the Lake Problem

Earlier described a more complicated model for pollution entering a lake with an oscillatory flow rate and an exponentially falling concentration of the pollutant entering the lake via the river.

- The initial value problem with \( c(0) = 5 = c_0 \)

\[
\frac{dc}{dt} = (0.01 + 0.005 \cos(0.0172t))(5e^{-0.002t} - c(t))
\]

- The Euler’s formula is

\[
c_{n+1} = c_n + h(0.01 + 0.005 \cos(0.0172t_n))(5e^{-0.002t_n} - c_n)
\]

- The model was simulated for 750 days with \( h = 1 \)
Graph of Simulation

Pollution in a Lake
Simulation: This solution shows a much more complicated behavior for the dynamics of the pollutant concentration in the lake.

- Could you have predicted this behavior or determined quantitative results, such as when the pollution level dropped below 2 ppm?
- This example is much more typical of what we might expect from more realistic biological problems.
- The numerical methods allow the examination of more complex situations, which allows the scientist to consider more options in probing a given situation.
- Euler’s method for this problem traces the actual solution very well, but better numerical methods are usually used.
Euler Example A: Consider the initial value problem

\[ \frac{dy}{dt} = -2y^2 \quad \text{with} \quad y(0) = 2 \]

- With a stepsize of \( h = 0.2 \), use Euler’s method to approximate \( y(t) \) at \( t = 1 \)
- Show that the actual solution of this problem is

\[ y(t) = \frac{2}{4t + 1} \]

- Determine the percent error between the approximate solution and the actual solution at \( t = 1 \)
Euler Example A

Solution: Euler’s formula with \( h = 0.2 \) for this example is

\[
y_{n+1} = y_n - h(2y_n^2) = y_n - 0.4y_n^2
\]

<table>
<thead>
<tr>
<th>( t_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 = 0 )</td>
<td>( y_0 = 2 )</td>
</tr>
<tr>
<td>( t_1 = t_0 + h = 0.2 )</td>
<td>( y_1 = y_0 - 0.4y_0^2 = 2 - 0.4(4) = 0.4 )</td>
</tr>
<tr>
<td>( t_2 = t_1 + h = 0.4 )</td>
<td>( y_2 = y_1 - 0.4y_1^2 = -0.4 - 0.4(0.16) = 0.336 )</td>
</tr>
<tr>
<td>( t_3 = t_2 + h = 0.6 )</td>
<td>( y_3 = y_2 - 0.4y_2^2 = 0.336 - 0.4(0.1129) = 0.2908 )</td>
</tr>
<tr>
<td>( t_4 = t_3 + h = 0.8 )</td>
<td>( y_4 = y_3 - 0.4y_3^2 = 0.2908 - 0.4(0.08459) = 0.2570 )</td>
</tr>
<tr>
<td>( t_5 = t_4 + h = 1.0 )</td>
<td>( y_5 = y_4 - 0.4y_4^2 = 0.2570 - 0.4(0.06605) = 0.2306 )</td>
</tr>
</tbody>
</table>
Euler Example A

Solution (cont): Verify that the solution is

\[ y(t) = \frac{2}{4t + 1} = 2(4t + 1)^{-1} \]

- Compute the derivative

\[ \frac{dy}{dt} = -2(4t + 1)^{-2}(4) = -8(4t + 1)^{-2} \]

- However, \(-2 (y(t))^2 = -2(2(4t + 1)^{-1})^2 = -8(4t + 1)^{-2}\)

- Thus, the differential equation is satisfied by the solution that is given

- At \( t = 1, y(1) = 0.4 \)

- The percent error is

\[ 100 \times \frac{y_{\text{Euler}}(1) - y_{\text{actual}}}{y_{\text{actual}}(1)} = \frac{100(0.2306 - 0.4)}{0.4} = -42.4\% \]
Euler Example B: Consider the initial value problem

\[ \frac{dy}{dt} = 2 \frac{t}{y} \quad \text{with} \quad y(0) = 2 \]

- With a stepsize of \( h = 0.25 \), use Euler’s method to approximate \( y(t) \) at \( t = 1 \)
- Show that the actual solution of this problem is
  \[ y(t) = \sqrt{2t^2 + 4} \]
- Determine the percent error between the approximate solution and the actual solution at \( t = 1 \)
Solution: Euler’s formula with \( h = 0.25 \) for this example is

\[
y_{n+1} = y_n + h \left( \frac{2t_n}{y_n} \right) = y_n + 0.5 \left( \frac{t_n}{y_n} \right)
\]

<table>
<thead>
<tr>
<th>( t_n )</th>
<th>( y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 = 0 )</td>
<td>( y_0 = 2 )</td>
</tr>
<tr>
<td>( t_1 = t_0 + h = 0.25 )</td>
<td>( y_1 = y_0 + 0.5t_0/y_0 = 2 + 0.5(0/2) = 2 )</td>
</tr>
<tr>
<td>( t_2 = t_1 + h = 0.5 )</td>
<td>( y_2 = y_1 + 0.5t_1/y_1 = 2 + 0.5(0.25/2) = 2.0625 )</td>
</tr>
<tr>
<td>( t_3 = t_2 + h = 0.75 )</td>
<td>( y_3 = y_2 + 0.5t_2/y_2 = 2.0625 + 0.5(0.5/2) = 2.1875 )</td>
</tr>
<tr>
<td>( t_4 = t_3 + h = 1.0 )</td>
<td>( y_4 = y_3 + 0.5t_3/y_3 = 2.1875 + 0.5(0.75/2) = 2.375 )</td>
</tr>
</tbody>
</table>
Euler Example B

Solution (cont): Verify that the solution is

\[ y(t) = (2t^2 + 4)^{0.5} \]

- Compute the derivative

\[ \frac{dy}{dt} = 0.5(2t^2 + 4)^{-0.5} (4t) = 2t(2t^2 + 4)^{-0.5} \]

- However, \( 2t/y(t) = 2t/(2t^2 + 4)^{0.5} = 2t(2t^2 + 4)^{-0.5} \)

- Thus, the differential equation is satisfied by the solution that is given

- At \( t = 1, \ y(1) = \sqrt{6} = 2.4495 \)

- The percent error is

\[ 100 \times \frac{y_{Euler}(1) - y_{actual}}{y_{actual}(1)} = \frac{100(2.375 - 2.4495)}{2.4495} = -3.04\% \]
Time-varying Population Model: A Malthusian growth model with a time-varying growth rate is

\[
\frac{dP}{dt} = (0.2 - 0.02t)P \quad \text{with} \quad P(0) = 5000
\]

- With a stepsize of \( h = 0.2 \), use Euler’s method to approximate \( P(t) \) at \( t = 1 \)
- Show that the actual solution of this problem is
  \[
  P(t) = 5000 e^{0.2t-0.01t^2}
  \]
- Determine the percent error between the approximate solution and the actual solution at \( t = 1 \)
- Use the actual solution to find the maximum population of this growth model and when it occurs
Solution: Euler’s formula with $h = 0.2$ for this example is

$$P_{n+1} = P_n + h(0.2 - 0.02t_n)P_n$$

<table>
<thead>
<tr>
<th>$t_n$</th>
<th>$P_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0 = 0$</td>
<td>$P_0 = 5000$</td>
</tr>
<tr>
<td>$t_1 = t_0 + h = 0.2$</td>
<td>$P_1 = P_0 + 0.2(0.2 - 0.02t_0)P_0 = 5200$</td>
</tr>
<tr>
<td>$t_2 = t_1 + h = 0.4$</td>
<td>$P_2 = P_1 + 0.2(0.2 - 0.02t_1)P_1 = 5403.8$</td>
</tr>
<tr>
<td>$t_3 = t_2 + h = 0.6$</td>
<td>$P_3 = P_2 + 0.2(0.2 - 0.02t_2)P_2 = 5611.35$</td>
</tr>
<tr>
<td>$t_4 = t_3 + h = 0.8$</td>
<td>$P_4 = P_3 + 0.2(0.2 - 0.02t_3)P_3 = 5822.3$</td>
</tr>
<tr>
<td>$t_5 = t_4 + h = 1.0$</td>
<td>$P_5 = P_4 + 0.2(0.2 - 0.02t_4)P_4 = 6036.6$</td>
</tr>
</tbody>
</table>
Solution (cont): Verify that the solution is
\[ P(t) = 5000 e^{0.2t-0.01t^2} \]

- Compute the derivative
  \[ \frac{dP}{dt} = 5000 e^{0.2t-0.01t^2} (0.2 - 0.02t) \]

- However, \((0.2 - 0.02t)P(t) = 5000 e^{0.2t-0.01t^2} (0.2 - 0.02t)\)
- Thus, the differential equation is satisfied by the solution that is given
- At \(t = 1\), \(P(1) = 6046.2\)
- The percent error is
  \[ 100 \times \frac{P_{Euler}(1) - P_{actual}}{P_{actual}(1)} = \frac{100(6036.6 - 6046.2)}{6046.2} = -0.16\% \]
Time-varying Population Model

Solution (cont): Maximum of the population

- The maximum is when the derivative is equal to zero
- Because $P(t)$ is positive, the derivative is zero (growth rate falls to zero) when $0.2 - 0.02t = 0$ or $t = 10$ years
- This is substituted into the actual solution

$$P(10) = 5000 e^1 = 13,591.4$$
**Improved Euler’s Method:** There are many techniques to improve the **numerical solutions of differential equations**

- Euler’s Method is simple and intuitive, but lacks accuracy
- Numerical methods are available through standard software, like Maple or MatLab
- Some of the best are a class of single step methods called **Runge-Kutta methods**
- The simplest of these is called the Improved Euler’s method
- Showing why this technique is significantly better than Euler’s method is beyond the scope of this course
Improved Euler’s Method Formula: This technique is an easy extension of Euler’s Method

- The Improved Euler’s method uses an average of the Euler’s method and an Euler’s method approximation to the function
- Let $y(t_0) = y_0$ and define $t_{n+1} = t_n + h$ and the approximation of $y(t_n)$ as $y_n$
- First approximate $y$ by Euler’s method, so define
  \[ y_{en} = y_n + h f(t_n, y_n) \]
- The Improved Euler’s formula starts with $y(t_0) = y_0$ and becomes the discrete dynamical system
  \[ y_{n+1} = y_n + \frac{h}{2} \left( f(t_n, y_n) + f(t_n + h, y_{en}) \right) \]
Example: Improved Euler’s Method: Consider the initial value problem:

\[
\frac{dy}{dt} = y + t \quad \text{with} \quad y(0) = 3
\]

- The solution to this differential equation is
  \[
y(t) = 4e^t - t - 1
  \]

- Numerically solve this using Euler’s Method and Improved Euler’s Method using \( h = 0.1 \)
- Compare these numerical solutions
Example: Improved Euler’s Method

Solution: Let $y_0 = 3$, the Euler’s formula is

$$y_{n+1} = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

The Improved Euler’s formula is

$$y_{e_n} = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

with

$$y_{n+1} = y_n + \frac{h}{2} \left( (y_n + t_n) + (y_{e_n} + t_n + h) \right)$$
$$y_{n+1} = y_n + 0.05 \left( y_n + y_{e_n} + 2t_n + 0.1 \right)$$
**Solution:** Below is a table of the numerical computations.

<table>
<thead>
<tr>
<th>$t$</th>
<th>Euler’s Method</th>
<th>Improved Euler</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$y_0 = 3$</td>
<td>$y_0 = 3$</td>
<td>$y(0) = 3$</td>
</tr>
<tr>
<td>0.1</td>
<td>$y_1 = 3.3$</td>
<td>$y_1 = 3.32$</td>
<td>$y(0.1) = 3.3207$</td>
</tr>
<tr>
<td>0.2</td>
<td>$y_2 = 3.64$</td>
<td>$y_2 = 3.6841$</td>
<td>$y(0.2) = 3.6856$</td>
</tr>
<tr>
<td>0.3</td>
<td>$y_3 = 4.024$</td>
<td>$y_3 = 4.0969$</td>
<td>$y(0.3) = 4.0994$</td>
</tr>
<tr>
<td>0.4</td>
<td>$y_4 = 4.4564$</td>
<td>$y_4 = 4.5636$</td>
<td>$y(0.4) = 4.5673$</td>
</tr>
<tr>
<td>0.5</td>
<td>$y_5 = 4.9420$</td>
<td>$y_5 = 5.0898$</td>
<td>$y(0.5) = 5.0949$</td>
</tr>
<tr>
<td>0.6</td>
<td>$y_6 = 5.4862$</td>
<td>$y_6 = 5.6817$</td>
<td>$y(0.6) = 5.6885$</td>
</tr>
<tr>
<td>0.7</td>
<td>$y_7 = 6.0949$</td>
<td>$y_7 = 6.3463$</td>
<td>$y(0.7) = 6.3550$</td>
</tr>
<tr>
<td>0.8</td>
<td>$y_8 = 6.7744$</td>
<td>$y_8 = 7.0912$</td>
<td>$y(0.8) = 7.1022$</td>
</tr>
<tr>
<td>0.9</td>
<td>$y_9 = 7.5318$</td>
<td>$y_9 = 7.9247$</td>
<td>$y(0.9) = 7.9384$</td>
</tr>
<tr>
<td>1</td>
<td>$y_{10} = 8.3750$</td>
<td>$y_{10} = 8.8563$</td>
<td>$y(1) = 8.8731$</td>
</tr>
</tbody>
</table>
Solution: Comparison of the numerical simulations

- It is very clear that the Improved Euler’s method does a substantially better job of tracking the actual solution.
- The Improved Euler’s method requires only one additional function, $f(t, y)$, evaluation for this improved accuracy.
- At $t = 1$, the Euler’s method has a $-5.6\%$ error from the actual solution.
- At $t = 1$, the Improved Euler’s method has a $-0.19\%$ error from the actual solution.